Extending Four Bugs on a Square: The Infinite Bug Chase

Joshua Briley (jsb22d@fsu.edu) & Dr. Bryan Quaife

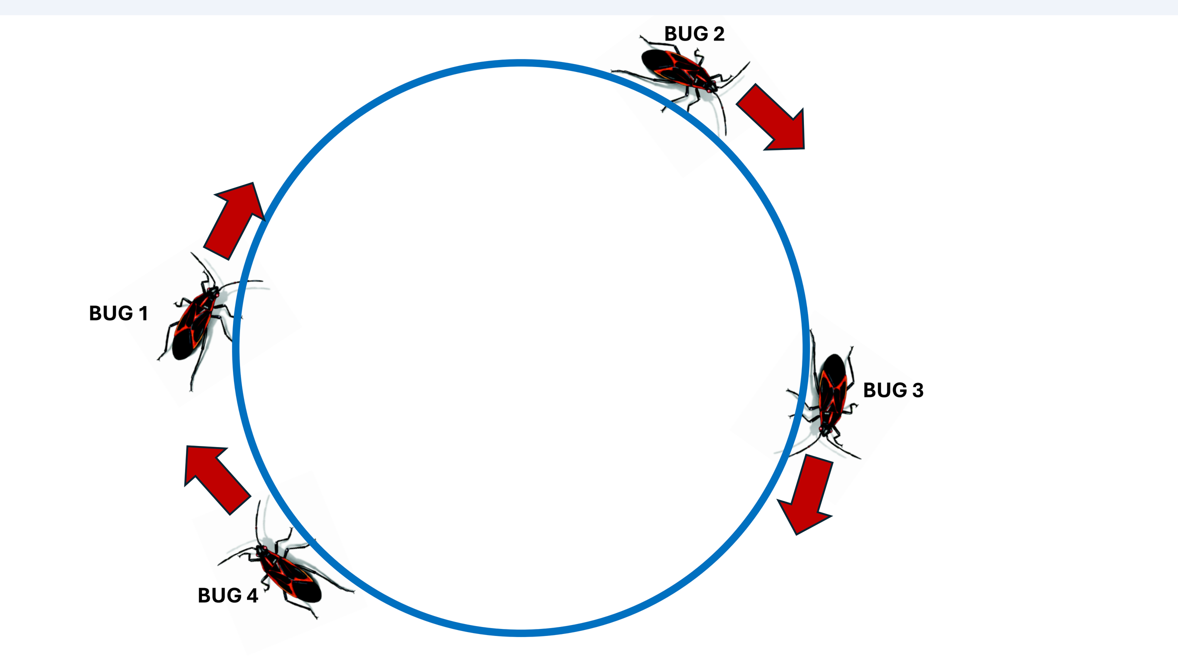
Abstract:

The classical problem of 'Four Bugs on a Square' is often used to motivate systems of differential equations. Initially positioned at the vertices of a square, the four bugs pursue one another in a cyclic fashion and with a constant speed. The bugs converge to one another, and the asymptotic behavior of this convergence is well-understood. An extension of this problem is to initialize and constrain the bugs to a surface. With this simple modification, interesting dynamics are revealed for geometries as simple as a circle.

When confined to a circle, the bugs eventually assume one of two states: convergence to a single point or an infinite chase loop. Using Monte Carlo methods, I estimate the probability that bugs initialized randomly will enter each of these two states. Notably, I show that the likelihood of the bugs converging to a point decreases as the number of bugs increases. I also show how probabilities can be calculated analytically, and I will out these calculations for the three-bug and four-bug scenarios.

Problem Formulation:

N bugs are randomly placed on the perimeter of a circle and assigned a number. Each bug chases the bug one number larger than itself at a uniform speed. This continues until bug N is chasing bug 1. Additionally, the bugs movement is limited only to the surface of the circle. With these conditions in place, the system can always be categorized into one of the following, either a limit cycle or a fixed point. The limit case refers to bugs that infinitely chase one another around the fixed point refers to bugs that eventually converge to a single point.



Motivation:

The classic 'Four Bugs on a Square' question has been thoroughly analyzed, but proof of further exploration into this question was demonstrated by Chapman and their analysis of 'Four Bugs on a *Rectangle*'. To take this question one step further we ask what happens if the bugs are constrained to the surface of a circle. Furthermore, the new system generalizes to larger number of bugs, and the trends that can be seen as the number of bugs increases.

Analysis:

Let $\theta_1, \ldots, \theta_N$ be the initial placement of the bugs. Define

$$\Xi_j = \{\theta_j | \; Bugs\ will\ reach\ a\ limit\ cycle\}$$

which depends on $\theta_1, \ldots, \theta_{j-1}$. We can assume $\Xi_1 = \{0\}$.

2 Bugs:

$$\Xi_1 = \{0\}, \Xi_2 = \{\pi\}$$

$$\mathrm{P}(\theta_2 \epsilon \Xi_2) = 0$$

3 Bugs:

$$\Xi_2 = (0,\pi), \Xi_3 = (\pi, \theta_2 + \pi)$$

$$\mathrm{P}(\theta_2 \epsilon\ \Xi_2 \cap \theta_3 \epsilon\ \Xi_3)$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_0^{\pi} \int_{\pi}^{\theta_2+\pi} d\theta_3\, d\theta_2 = \ \ldots = \frac{1}{8}$$

$$\therefore P(Limit\ Cycle\ \cap\ \theta_2 < \theta_3) = \frac{1}{8}$$

$$\therefore P(Limit\ Cycle\ ) = \frac{1}{4}$$

4 Bugs:

$$\Xi_2 = (0,\pi), \Xi_3 = (\pi, \theta_2 + \pi),$$

$$\Xi_4 = (\max(\theta_3, \pi)\,, \min((\theta_3 + \pi, 2\pi))$$

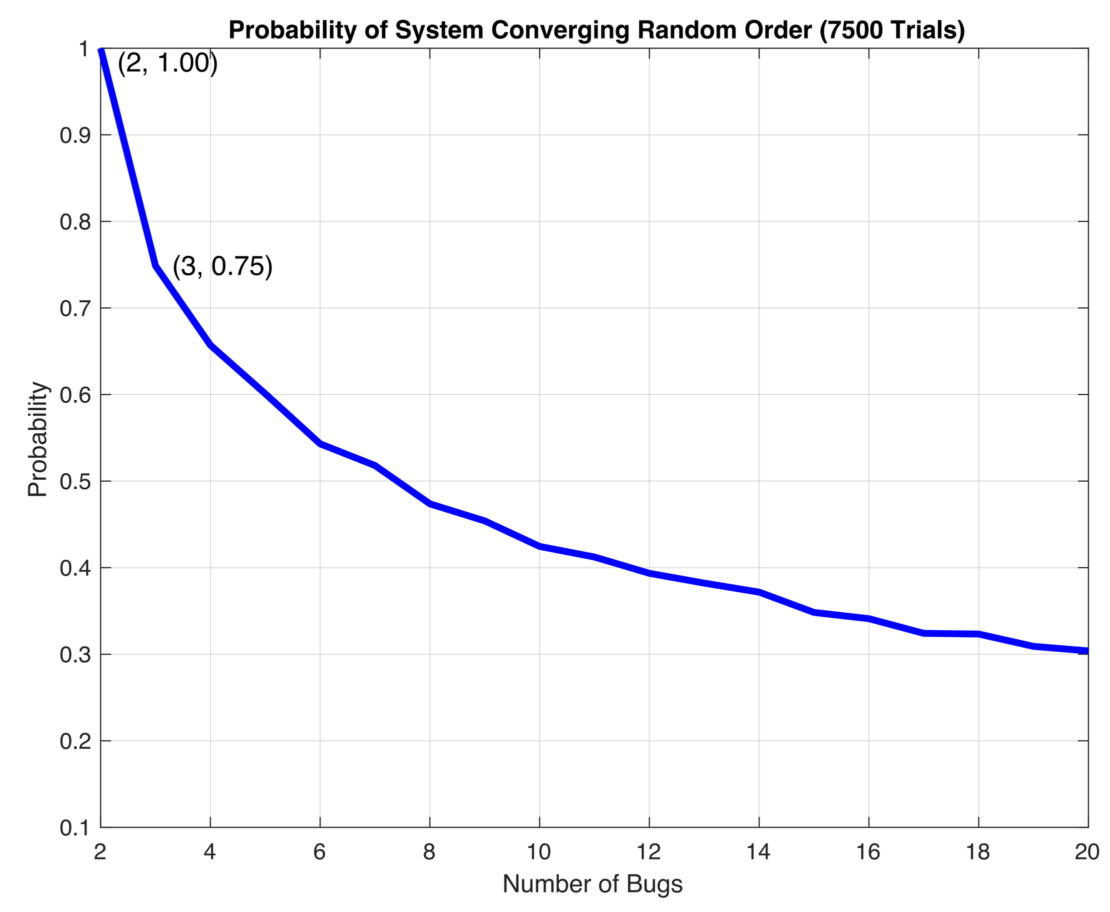
$$\mathrm{P}(\theta_2 \epsilon\ \Xi_2 \cap \theta_3 \epsilon\ \Xi_3 \cap\ \theta_4\ \epsilon\ \Xi_4) =$$

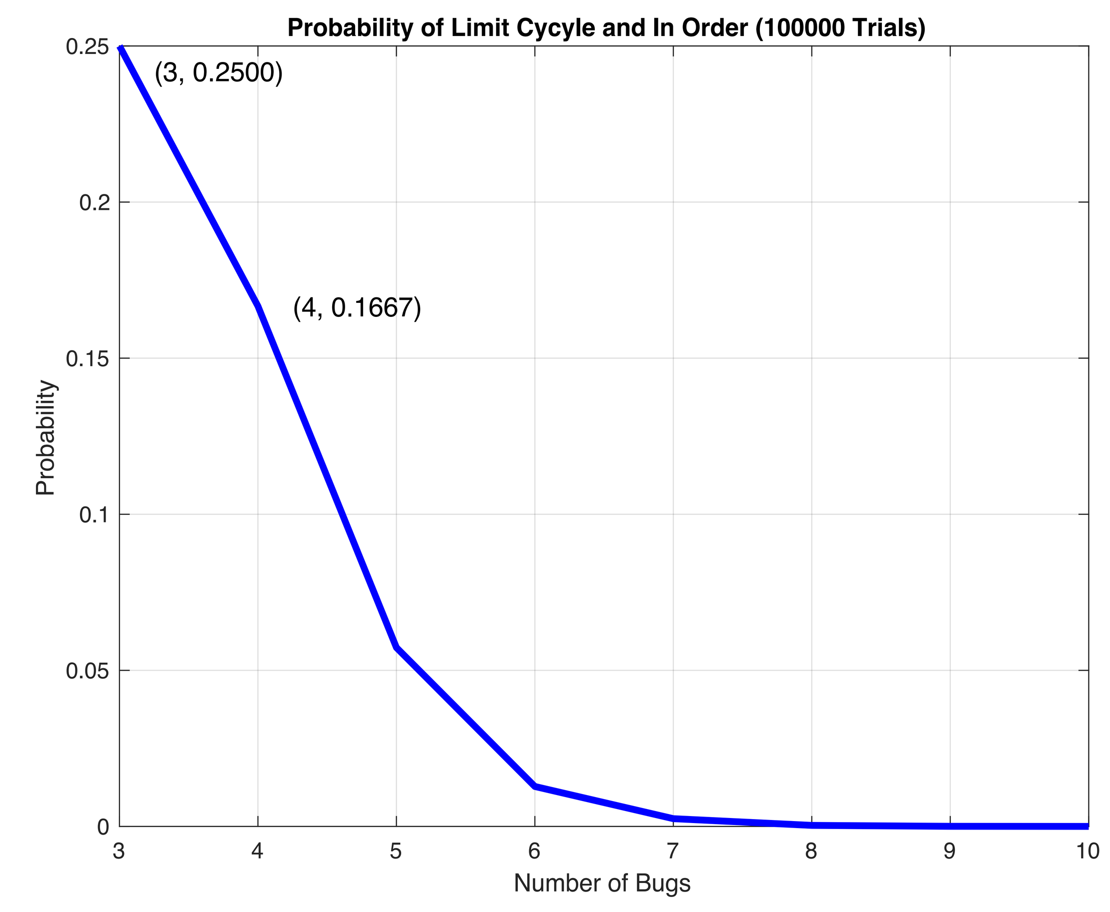
$$\left(\frac{1}{2\pi}\right)^2 \int_0^{\pi} \left[\int_{\theta_2}^{\pi} \int_{\pi}^{\theta_3} d\theta_4 d\theta_3 + \int_{\pi}^{\theta_2+\pi} \int_{\pi}^{2\pi} d\theta_4 d\theta_3\right] d\theta_2 = \frac{1}{12}$$

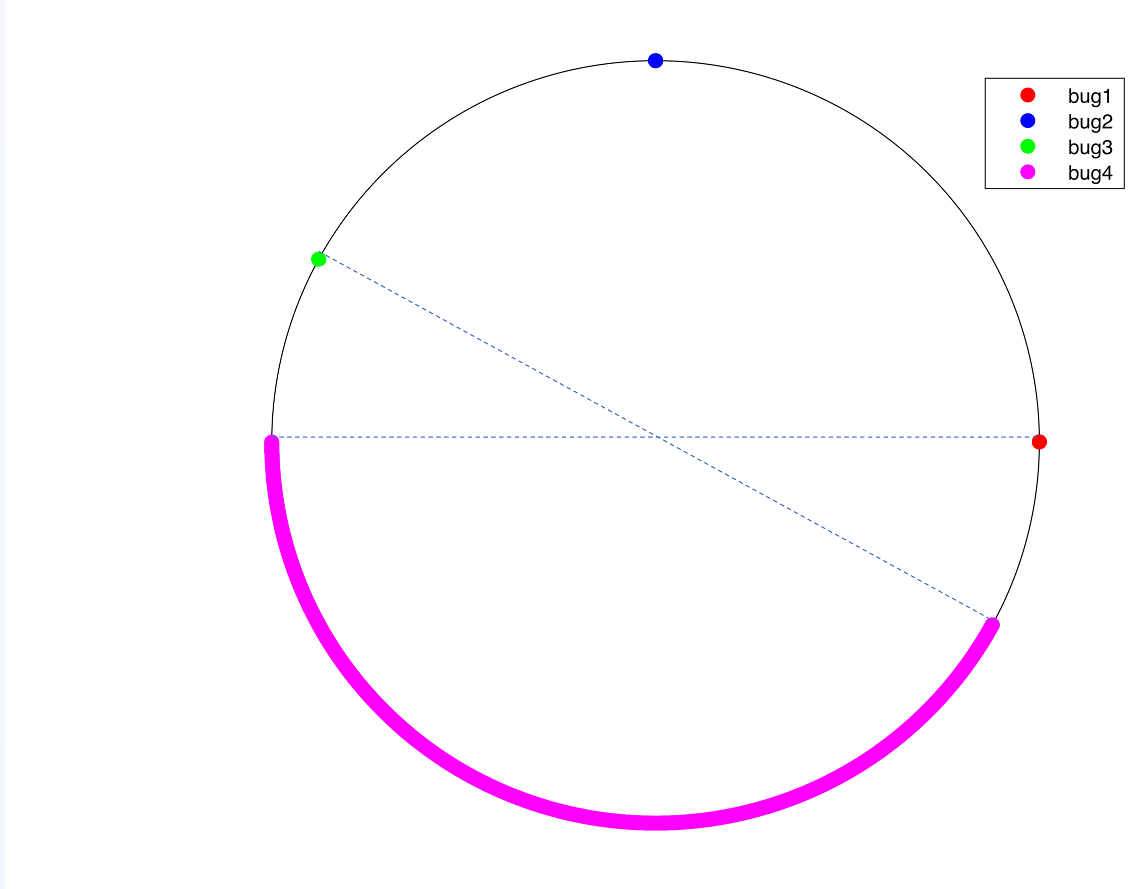
$$\therefore P(Limit\ Cycle\ \cap\ \theta_2 < \theta_3 < \theta_4) = \frac{1}{12}$$

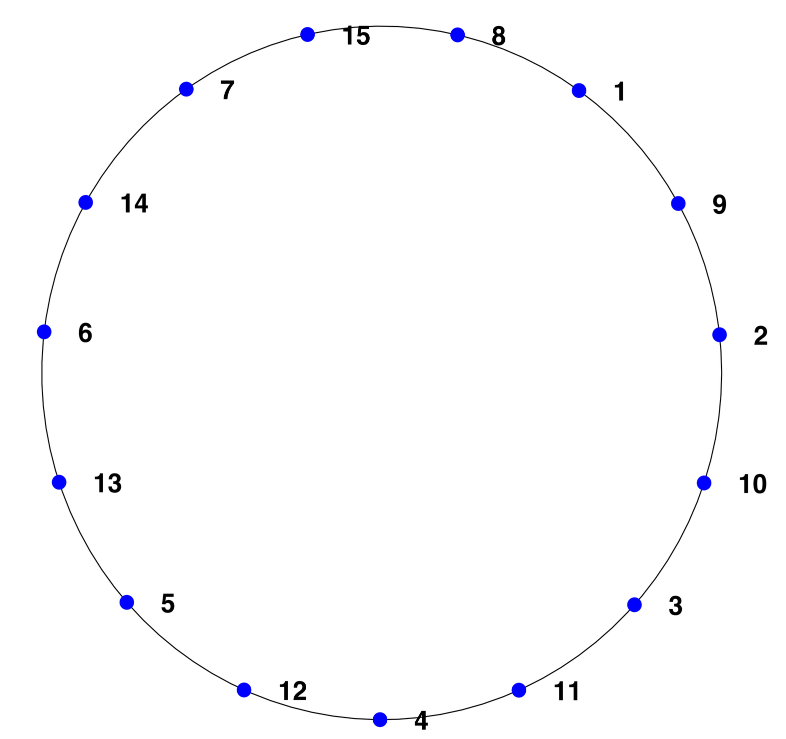
$$\therefore P(Limit\ Cycle \cap Initially\ in\ Order) = \frac{1}{6}$$

Monte Carlo Simulations









Future Work:

Future work includes a continued analysis for a greater number bugs as well as analyzing dynamics of bugs on other 2-dimesnsional and 3-dimensional surfaces such as a torus.

References:

Chapman S.J Lottes James and Trefethen Lloyd N. 2011Four bugs on a rectangleProc. R. Soc. A.467881896

Antonick, G. (2014, September 8). *Bugs on a square*. The New York Times. https://archive.nytimes.com/wordplay.blogs.nytimes.com/2014/09/08/bugs/

Discussion

As the number of bugs increases, it becomes more likely that the case seen to the right will occur. In this case the bug index spans larger than the perimeter of circle. Instead of the typical case where the bugs are in order, counting counterclockwise we can see each bug has mod15(N + 8) next to it, followed by N + 1.