A Multiscale Implementation for a Nonlocal Model of Mechanics

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The Nonlocal Model [1, 2]

Nonlocal Boundary Value Problem

- The *nonlocal volume-constrained problem* is given by

$$\begin{cases} -\mathcal{L}(u(\mathbf{x})) = b(\mathbf{x}), & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \Omega_{\mathcal{I}} \end{cases}$$

where $\mathcal{L}u = 2\int_{\mathbb{R}^n} (u(\mathbf{x}') - u(\mathbf{x}))\gamma(\mathbf{x},\mathbf{x}')d\mathbf{x}'$, $\gamma(\mathbf{x},\mathbf{x}') = \boldsymbol{\alpha}(\mathbf{x},\mathbf{x}') \cdot (\boldsymbol{\Theta}(\mathbf{x},\mathbf{x}') \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{x}'))$, $\boldsymbol{\Theta}$ denotes a second-order, positive definite symmetric tensor, $\boldsymbol{\alpha}(\mathbf{x},\mathbf{x}') : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given anti-symmetric mapping, i.e.,$\boldsymbol{\alpha}(\mathbf{x},\mathbf{x}') = -\boldsymbol{\alpha}(\mathbf{x}',\mathbf{x})$.

Peridynamics is a nonlocal model

- Nonlocal vector calculus can be applied to the analysis of the peridynamic model for any dimension. In other words, *the peridynamic model is a special case of the nonlocal models*.
- To take the one-dimensional case as a simple example, by setting

$$\gamma(x,x') = \begin{cases} \dfrac{c}{2|x-x'|}, & \text{if } x \in H_{\mathbf{x}}, \\ 0, & \text{otherwise,} \end{cases}$$

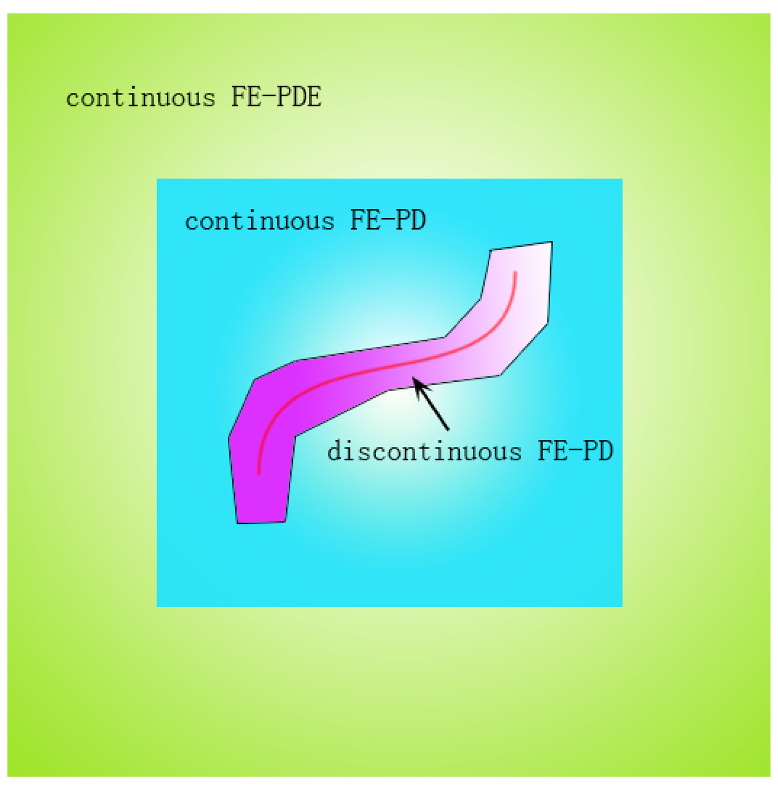
the nonlocal diffusion equation $2\int_{\Omega}(u(\mathbf{x}) - u(\mathbf{x}'))\gamma(\mathbf{x},\mathbf{x}')d\mathbf{x}' = b(\mathbf{x})$ becomes the peridynamic equation of motion $\int_{H_{\mathbf{x}}} \dfrac{u(x) - u(x')}{|x-x'|}dx' = b(x)$, where $H_{\mathbf{x}}$ is a neighborhood of the particle located at point x.

A Multiscale Implementation

Peridynamics Is a Multiscale Model

- The multiscale nature of peridynamics is revealed by changing the local grid size h_{local}:
  - if $h_{local} \ll \delta$, peridynamics is decidedly nonlocal;
  - if $h_{local} \gg \delta$, and the displacement field is smooth, peridynamcis reduces to a local model.
- Thus, by just changing the grid size, peridynamics is a multiscale model.

Framework & Illustration



- red curve: discontinuity occurs
- purple region: use an anisotropic grid – fine grid spacing across the discontinuity (the red curve), no refinement "parallel" to the discontinuity
- blue region: use an unrefined grid for transition from discontinuous to continuous finite elements
- green region: use standard FE methods for PDEs on an unrefined grid

Multiscale finite element discretization of peridynamics

1. detect the triangles that contain the discontinuity;
2. make grid refinement near discontinuity;
3. for elements containing the discontinuity, use discontinuous Galerkin (DG) methods for peridynamics;
4. for the remaining elements, used an unrefined grid with perhaps DG for peridynamics near the discontinuity and continuous Galerkin (CG) methods for peridynamics away from the discontinuity ;
5. use quadrature rules that work for any $\delta - h$ combination.
6. further away from the discontinuity, switch to standard finite element/partial differential equation discretizations.

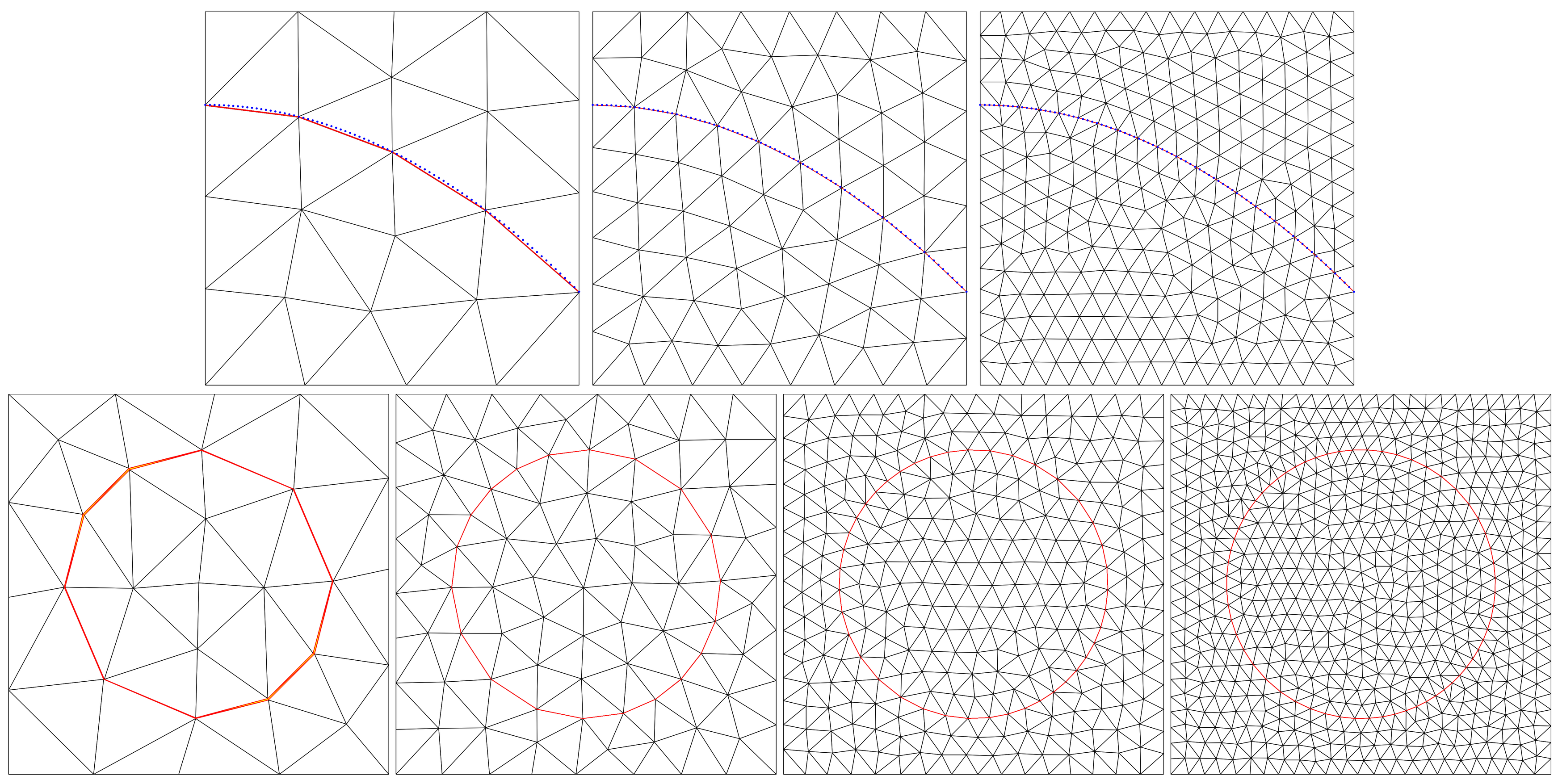
Local Grid Refinement

Motivation from 1D results [3]

- We use the Galerkin Finite Element Method to solve the problem and two finite element spaces are selected: space of continuous piecewise linear functions (CL) and space of discontinuous piecewise linear functions (DL). Suppose the exact solution has a jump discontinuity at a point in 1D.
- If a grid point is located at the point of discontinuity, $\|u - u^h\|_{L^2(\Omega')} = O(h^2)$ for DL and $\|u - u^h\|_{L^2(\Omega')} = O(h^{1/2})$ for CL. If no grid point is located at the point of discontinuity, $\|u - u^h\|_{L^2(\Omega')} = O(h^{1/2})$ for both DL and CL.
- However, if one does abrupt local refinement with an element of width h^4 surrounding the discontinuity, then for DL, $\|u - u^h\|_{L^2(\Omega')} = O(h^2)$ and, if one excludes the elements containing the discontinuity, $\|u - u^h\|_{L^\infty(\Omega')} = O(h^2)$ as well.
- Combine the advantages of CL and DL method, we can do an abrupt local refinement $h \to h^4$, impose DL on the discontinuous intervals/elements, and CL on other intervals/elements.

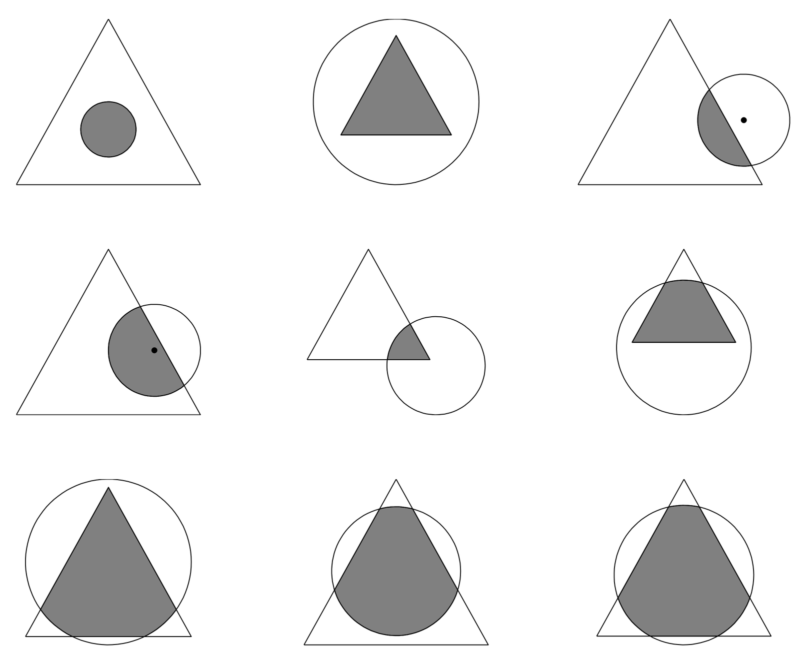
Goals of Refinement Strategy in 2D

1. Refine locally to get a thin layer of elements containing the discontinuity.
   - the elements in the layer should have thickness $O(h^4)$ across the layer.
   - but they should be $O(h)$ parallel to the layer.
2. The elements outside the layer should not be thin and have linear dimension $O(h)$.
3. The transition between the elements in the layer to those outside the layer should be abrupt.
   - no transition zone from thin to regular elements



Quadrature Issue

- When evaluating the double integrals in the weak formulation of the nonlocal problem, the integral domain for the inner integral is an intersection of a triangular element and a disk.
- There are several situations of the shapes of the intersections due to the relative position of the triangle and the disk (see right figures), which makes it difficult to deal with.



- One can observe in all situations that the intersection part is a union of several small triangles and circular segments. Once we know the quadrature rules on triangles and circular segments, the problem will be solved. There are many quadrature rules for triangles available. And fortunately, we found a paper giving a Guassian quadrature rule on circular segments, see [4].

References

[1] Q. Du, M. Gunzburger, R. Lehoucq, K. Zhou, *A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws*, Mathematical Modeling and Methods in Applied Science, Volume 23, 2013, 493-540.

[2] Q. Du, M. Gunzburger, R. Lehoucq, Kun Zhou, *Analysis and approximation of nonlocal diffusion problems with volume constraints*, SIAM Review, Volume 54, Number 4, 667-696, 2012.

[3] X. Chen, M. Gunzburger, *Continuous and discontinuous finite element methods for a peridynamics model of mechanics*, Computer Methods in Applied Mechanics and Engineering, Volume 200, 2011, 1237-1250.

[4] G. Fies and M. Vianello, *Trigonometric gaussian quadrature on subintervals of the period.* Electronic Transactions on Numerical Analysis, 39:102-112, 2012.