Mean Motions in the *Planetary Hypotheses*

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The first part of Ptolemy's *Planetary Hypotheses* (hereinafter *PH*),[1] and the only part surviving in Greek, is described by Ptolemy as a succinct summary of the planetary models in the *Almagest*.[2] He says his summary is arranged so that the models may be more easily understood by people 'like ourselves', presumably people knowledgeable in theoretical astronomy, and by people who build mechanical models.

If this was all we learned from the *PH* then it would hardly be worth writing about, since we have the *Almagest* and it explains the planetary models in great detail. But what makes this part of the *PH* interesting is that while Ptolemy generally follows the *Almagest,* e.g. he explains the models in the same order: Sun, Moon, Mercury, Venus, Mars, Jupiter, Saturn, and he uses the same model structures: an eccentre for the Sun, crank mechanisms plus epicycle for the Moon and Mercury, and the equant plus epicycle for Venus, Mars, Jupiter, and Saturn, when it comes to details we find many differences with respect to the *Almagest*.

Now Ptolemy in fact remarks that he has made many corrections to his models 'on the basis of more prolonged comparisons of observations'– corrections to the model structures, to the relative size of various model elements, and to the periodic motions. Indeed we do find such changes: the model for the Moon and the latitude models for the inner and outer planets, the radius of the crank circle for Mercury and possibly the eccentricity for Saturn, and all of the mean motions except those for the fixed stars and the Sun are different from what is found in the *Almagest*.

In the following the technical content of the first part of the *PH* will be reviewed, with particular attention to the mean motions and the associated epoch values.[3]

A. Simple Periods from Almagest corrections

Each of Ptolemy's planetary models has one or more angles that increase uniformly with time. The rate of increase is called a mean motion, in contrast to the true motion which is generally not uniform. Mean motions are empirically determined by observing how long it takes for an angle to advance by some amount, and the accuracy of the mean motion is improved as the time baseline lengthens. In most cases the angles that are increasing uniformly are not directly observable, and much of the *Almagest* is devoted to explaining how to deduce the values of these mean angles from observations of planetary positions that are increasing nonuniformly due to the various anomalies and which are generally not at integral numbers of return in the mean angle. These deductions, however, depend on the values of previously determined model parameters such as the eccentricity and epicycle radius and the direction of the apsidal line, and changes in these values will, in general, lead to changes in the mean angles and hence the derived mean motions.

In almost all cases mean motions are summarized as *period relations*: statements that some number of returns or restitutions in some angle occur in some number of years. With enough accurate empirical data, it is even possible to specify small adjustments to the number of returns and years, so that so-called ‘corrected’ period relations more closely approximate the underlying empirical data. These period relations, corrected or not, need not necessarily lead to mean motions that agree exactly with those found from observations separated by long intervals of time, typically several to many centuries, and the question of which comes first, the long time interval determination, or successively refined period relations estimated from observed returns, or perhaps some iteration using both, is not necessarily clear in the historical records we have.

Regarding the periodic restitutions, Ptolemy writes

> Now that these things have been outlined, next let us go on to the models of the planets, setting out first their simple and unmixed periods, out of which the particular, complex ones arise; these [simple and unmixed periods] were obtained by us as approximations to the restitutions computed from the correction.

The ‘simple and unmixed’ periods that Ptolemy is referring to correspond to the mean motion of the fixed stars and the planetary apogees and nodes, the Sun, the planets with respect to the Sun (elongation for the Moon and anomaly for the five planets), plus the mean motions in anomaly and latitude for the Moon.

Ptolemy’s numerical values are as follows:

In 300^{y} (Egyptian years of 365^{d}) and 74^{d} the sphere of the fixed stars as well as the apogees and nodes of the five planets make $1/120^{th}$ return relative to the tropical and equinoctial points of the zodiac.

In the same period, the Sun makes 300 returns relative to the tropical and equinoctial points of the zodiac.

In 8523 tropical years the Moon makes 105416 returns to the Sun (synodic months), in 3277 months it makes 3512 returns in anomaly, and in 5458 months it makes 5923 returns in latitude.

Finally, Ptolemy specifies, for each of the five planets, the number of sidereal years required to complete some number of returns in anomaly:

	sidereal years	*returns in anomaly*
Mercury	993	3130
Venus	964	603

Mars	1010	473
Jupiter	671	603
Saturn	324	303

Our immediate task is to understand how Ptolemy arrived at these numbers as 'approximations to the restitutions computed from the correction'. As a first and certainly most obvious assumption, let us suppose that 'the restitutions computed from the correction' are simply those given in the *Almagest*.

Some of the *PH* period relations agree exactly with the corresponding parts of the *Almagest*. The statement that 300 tropical years takes $300^y\ 74^d$ follows exactly from the length of the Hipparchan solar year of $365 + \frac{1}{4} - \frac{1}{300}{}^d$ used in all of Ptolemy's astronomical works. Since 1/120th return is just 3° in 300^{ty}, the speed of the fixed stars and the five apogees corresponds to a precession constant of 1° per 100^{ty} and over even a millennium timescale is negligibly different from the precession constant of 1° in 100^y that Ptolemy specifies in the *Canobic Inscription* and uses in the *Almagest*. And as Ptolemy helpfully remarks, it also follows that 36,000 tropical years is exactly equal to $36{,}024^y\ 120^d$ and 35,999 sidereal years. If Ptolemy is in fact referring to 'approximations to the restitutions computed from the correction' for these motions, it is presumably the 1/300th of a day correction to the year and the small inequality of the tropical and sidereal years that he is referring to.

Regarding the lunar mean motions Ptolemy devotes most of *Almagest* IV to explaining and then 'correcting' a set of period relations for lunar motion that he says Hipparchus had "proved, by calculations from observations made by the Chaldeans and in his time". Hipparchus' results are that the Moon makes

1) 4267 returns in elongation from the Sun in $126007^d\ 1^h$, which gives a synodic month of approximately $29;31,50,8,20^d$, and which combined with Hipparchus' tropical year length leads to a lunar mean motion in longitude of $13;10,34,58,33,30,30^{\circ/d}$;

2) 4573 returns in anomaly in 4267 months (equivalent to 269 returns in anomaly in 251 months), which leads to a mean motion in anomaly of $13;3,53,56,29,38,38^{\circ/d}$;

3) 5923 returns in latitude in 5458 months, which leads to a mean motion in latitude of $13;13,45,39,40,17,19^{\circ/d}$.

Then in *Almagest* IV 7 and 9 Ptolemy 'corrects' the mean motions that result from these period relations by examining pairs of eclipses separated by over 800 years. The result of these corrections is that the mean motion in longitude is unchanged, the mean motion in anomaly is reduced by $0;0,0,0,11,46,39^{\circ/d}$, and the mean motion in latitude is increased by $0;0,0,0,8,39,18^{\circ/d}$. In fact, in *Almagest* IV 9 Ptolemy is even correcting an earlier correction to the mean motion in latitude that he included in the *Canobic Inscription*.[4]

Inspecting the period relations Ptolemy gives, we realize that they can all be built from combinations of shorter, well known relations:

$$\frac{105416}{8523} = \frac{448 \cdot 235 + 136}{448 \cdot 19 + 11}$$

$$\frac{3512}{3277} = \frac{13 \cdot 269 + 15}{13 \cdot 251 + 14}$$

$$\frac{5923}{5458} = \frac{7 \cdot 777 + 2 \cdot 242}{7 \cdot 716 + 2 \cdot 223}$$

Thus suggests that Ptolemy might have derived the *PH* relations from corrected mean motions known earlier, using a theorem apparently well-known at the time, that if

$$\frac{a}{b} < \frac{c}{d}$$

then

$$\frac{a}{b} < \frac{ma + nc}{mb + nd} < \frac{c}{d}$$

for any positive integers m and n.[5] So let us suppose that we have some corrected mean motion ω, and that we know upper and lower limits on ω from familiar period relations, i.e.

$$\frac{a}{b} < \omega < \frac{c}{d}.$$

It follows that we also know the errors ε and ε' determined by

$$\omega = \frac{a}{b} + \varepsilon$$

and

$$\omega = \frac{c}{d} - \varepsilon'$$

and our task is to find values of m and n that make $\frac{ma + nc}{mb + nd}$ as close as possible to ω. It is easy to show that this is accomplished by picking m and n according to

$$\frac{n}{m} \simeq \frac{b\varepsilon}{d\varepsilon'}.$$

It is also the case that if one constructs a continued fraction expansion of ω, and if the approximations *a/b* and *c/d* occur in the expansion, then the approximation $\frac{ma+nc}{mb+nd}$ with optimal values of *m* and *n* will also occur in the expansion.

Let us see how Ptolemy might have applied this to the problem of approximating the corrected mean motions of the moon. Using Hipparchus' values he would find the number of months per tropical year as

$$\omega = \frac{4267^{m}}{126{,}007\tfrac{1}{24}^{d} / 365;14,48^{d/ty}} = 12;22,6,17,23^{m/ty}$$

corresponding to the month length 29;31,50,8,9^{d}. He would certainly have been well aware of the shorter period relations, 136^{m} in 11^{y} and 235^{m} in 19^{y}, which bound this value of ω, and these lead to the errors

$$\varepsilon \simeq 0;0,17,12,26^{m/ty}$$

and

$$\varepsilon' \simeq 0;0,0,1,4^{m/ty}$$

leading to an estimate for the optimal *n/m* value of about 560-562, depending on how one does the rounding. Using instead the System B synodic month of 29;31,50,8,20^{d}, the only month actually used in the *Almagest*, one finds the optimal *n/m* value of about 525-527 (exact arithmetic gives 525):

$$\frac{136}{11} < \frac{525\cdot 235+136}{525\cdot 19+11} = \frac{123511}{9986} < \frac{235}{19}$$

However, what Ptolemy in fact used was *n/m* = 448, and to get that value for *n/m* using the method we are discussing or, equivalently, to find the ratio 105416/8523 in the continued fraction expansion based on his month length, he would have had to start with a synodic month length in the range 29;31,50,8,37–29;31,50,8,52 (his ratio gives 29;31,50,8,48). This suggests that either Ptolemy was not working in this direction at all and hence was using a poor approximation, or that he was working in this direction but was using a month length with a fourth digit in the range 37-52.

Similarly, if Ptolemy was using the *Almagest* corrected mean motion in anomaly ω_a = 13;3,53,56,17,51,59$^{\circ/d}$, which leads to 1;4,18,10,1,25,28$^{r/m}$, he would have found an optimal *n/m* of about 213, whereas in fact he uses 13, which leads to 1;4,18,9,46,38,3$^{r/m}$, and which would follow from combining the *PH* month[6] with any ω_α in the range 13;3,53,52,58$^{\circ/d}$ – 13;3,53,53,18$^{\circ/d}$, significantly below the corrected *Almagest* value.

Moreover, since the corrected value in the *Almagest* is explicitly constructed by adding a very small increment to the value resulting from the basic 269 returns in 251 months, it is hard to understand why Ptolemy did not simply give 269 returns in 251 months, or the equivalent 4573 returns in 4267 months, as his *PH* approximation, since it is the starting value for the correction in the *Almagest* and leads to the value 1;4,18,10,2,23,26$^{r/m}$ which is a significantly better approximation to the corrected *Almagest* value than the 3512 returns in 3277 months relation he gives. Once again this suggests that if he was making a good approximation, he was not starting with the corrected *Almagest* value.

Finally, the corrected value for the mean motion in latitude in the *Almagest* is explicitly constructed by adding a very small increment to the value resulting from the Hipparchan 5923 returns in 5458 months, and this is the *PH* ratio. That he did not do the same thing in the parallel case of the lunar anomaly, however, suggests that we at least probe a little deeper. Now the *Almagest* corrected mean motion in latitude, ω_b = 13;13,45,39,48,56,38$^{\circ/d}$, may be converted into returns per month using the *PH* synodic month, yielding 1;5,6,42,22,27,13$^{r/m}$ and the continued fraction expansion of this gives the following sequence of convergents:

$$12/11, 13/12, 38/35, 51/47, 242/223, 777/716, 1796/1655, 2573/2371, 6942/6397, \ldots$$

that does not include 5923/5458. In fact, the error using 5923/5458 is more than twice as large as the error from the smaller ratio 2573/2371 that does occur in the series of convergents. Even if Ptolemy was using the *Almagest* synodic month to convert the daily speed into revolutions per month, 5923/5458 still does not occur in the series of convergents, and the error using 5923/5458 is only about 30% smaller than the error from the smaller ratio 2573/2371 that once again occurs in the sequence of convergents.

If Ptolemy was indeed not starting from the corrected *Almagest* values, but was trying to make an accurate approximation, we can work backwards and ask what range of values for the mean motions will have the simple *PH* relations appear in their continued fraction expansion. The results are shown in Table 1.

Table 1

	Lower limit	Upper limit	*Almagest*
elongation	12;11,26,41,8,37$^{\circ/d}$	12;11,26,41,8,52$^{\circ/d}$	12;11,26,41,20$^{\circ/d}$
anomaly	13;3,53,52,41	13;3,53,53,18	13;3,53,56,18
latitude	13;13,45,39,25	13;13,45,39,31	13;13,45,39,49

In each case the *Almagest* value lies well outside the allowed interval, which suggests that for the Moon Ptolemy was starting with mean motions different from those found in the *Almagest*. The modern values for these mean motions in A.D. 150, corrected for ΔT, are 12;11,26,41,20$^{\circ/d}$, 13;3,53,55,43$^{\circ/d}$, and 13;13,45,39,2$^{\circ/d}$, respectively.

In *Almagest* IX 3 the mean motions in anomaly of the five planets are derived from finely-tuned period relations. Ptolemy explains that Hipparchus had computed for each planet "the smallest period in which it makes an approximate return in both anomalies" and that "These [periods] have been corrected by us, on the basis of the comparison of their positions which became possible after we had demonstrated their anomalies, as we shall explain at that point [in *Almagest* IX 10, X 4, X 9, XI 3, and XI 7]". Unlike the cases with the Moon, however, Ptolemy does not tell us exactly what period relations Hipparchus used, and so we do not known what Ptolemy was 'correcting'. In any event, the mean motions in anomaly that Ptolemy gives in the *Almagest* were ultimately derived as follows:[7]

Saturn $$\omega_a = \frac{57^r}{59^{ty} + (1 + \tfrac{1}{2} + \tfrac{1}{4})^d} \simeq \frac{20520^\circ}{21551;18^d} = 0;57,7,43,41,43,40^{\circ/d}$$

Jupiter $$\omega_t = \frac{6^r - (4 + \tfrac{1}{2} + \tfrac{1}{3})^\circ}{71^{ty} - (4 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{15})^d} \simeq \frac{2155;10^\circ}{25927;37^d} = 0;4,59,14,26,46,31^{\circ/d}$$
$$\omega_a = \omega_S - \omega_t = 0;54,9,2,46,26,0^\circ$$
$$\neq \frac{65^r}{71^{ty} - (4 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{15})^d} \simeq \frac{23400^\circ}{25927;37^d} = 0;54,9,2,42,55,52^{\circ/d}$$

Mars $$\omega_t = \frac{42^r + (3 + \tfrac{1}{6})^\circ}{79^{ty} + (3 + \tfrac{1}{6} + \tfrac{1}{20})^d} \simeq \frac{15123;10^\circ}{28857;43^d} = 0;31,26,36,53,51,33^{\circ/d}$$
$$\omega_a = \omega_S - \omega_t = 0;27,41,40,19,20,58^\circ$$
$$\neq \frac{37^r}{79^{ty} + (3 + \tfrac{1}{6} + \tfrac{1}{20})^d} \simeq \frac{13320^\circ}{28857;43^d} = 0;27,41,40,11,44,37^{\circ/d}$$

Venus $$\omega_a = \frac{5^r}{8^{ty} - (2 + \tfrac{1}{4} + \tfrac{1}{20})^d} \simeq \frac{1800^\circ}{2919;40^d} = 0;36,59,25,53,11,28^{\circ/d}$$

Mercury $$\omega_a = \frac{145^r}{46^{ty} + (1 + \tfrac{1}{30})^d} \simeq \frac{52200^\circ}{16802;24^d} = 3;6,24,6,59,35,50^{\circ/d}$$

In the derivations for Jupiter and Mars, $\omega_S = \frac{360^\circ}{365 + \tfrac{1}{4} - \tfrac{1}{300}{}^d} = 0;59,8,17,13,21,31^{\circ/d}$ is the mean motion in tropical longitude of the Sun, and although Ptolemy claims that the mean motions he gives follow directly from the revolutions in anomaly, as in the third lines, in fact they follow indirectly from the mean motions in longitude, as shown in the first and second lines.

Just as we saw for the lunar month and anomaly, the 'simple and unmixed' *PH* period relations that Ptolemy uses are simple combinations of well-known shorter relations:

Saturn $$\frac{313}{324} = \frac{5 \cdot 57 + 28}{5 \cdot 59 + 29}$$

Jupiter $\frac{706}{771} = \frac{391+315}{427+344}$

Mars $\frac{473}{1010} = \frac{303+170}{647+363}$

Venus $\frac{603}{964} = \frac{3\cdot152+29\cdot5+2}{3\cdot243+29\cdot8+3}\left(=\frac{4\cdot152-5}{4\cdot243-8}\right)$

Mercury $\frac{3130}{993} = \frac{2\cdot1223+684}{2\cdot388+217}$

All of the ratios on the right hand sides of the above equations occur in the continued fraction expansions of the corrected mean motions given in the *Almagest*, but only the *PH* ratios for Saturn and Mercury are optimal in the sense of continued fractions – no smaller fraction is a better approximation. However, for Jupiter Ptolemy's choice of 706/771 is actually inferior to 391/427, for Mars his choice of 473/1010 is inferior to 303/647, and for Venus his 603/964 is inferior to 152/243. In all three cases the error for Ptolemy's choice is about three times larger than the error from the shorter period relation he could have used. This strongly suggests that, at least for Jupiter, Mars and Venus, Ptolemy was finding an approximation to something other than the mean motion values we find in the *Almagest*.

We can again investigate this further by working backward and asking how much we can tinker with the *Almagest* mean motions and still have the simple *PH* relations appear in the continued fraction expansion. The results are shown in Table 2.

Table 2

	Lower limit	Upper limit	*Almagest*
Saturn	0;57,7,43,31$^{\circ/d}$	0;57,7,45,13$^{\circ/d}$	0;57,7,43,41$^{\circ/d}$
Jupiter	0;54,9,3,3	0;54,9,3,31	0;54,9,2,46
Mars	0;27,41,40,28	0;27,41,40,44	0;27,41,40,19
Venus	0;36,59,27,19	0;36,59,27,29	0;36,59,25,53
Mercury	3;6,24,6,58	3;6,24,7,15	3;6,24,6,59

For Jupiter, Mars, and Venus the *Almagest* value lies well outside the allowed interval, and even for Saturn and Mercury the *Almagest* value lies barely inside the allowed interval (relative to the width of that interval).

Altogether then, for both the Moon and the planets, there is certainly no significant evidence that Ptolemy was diligently trying to find good approximations to the 'restitutions computed from the correction' as we know them from the *Almagest*. Assuming that he was trying to make good approximations, we have found the intervals in which his underlying values must have fallen.

B. Complex Periods from Simple Periods

After giving the simple period relations discussed above, Ptolemy proceeds to a series of discussions of the model for each individual planet and as part of each such discussion he gives what he calls the ‘particular, complex’ period relations, which he says arise from the simple relations.

For the Sun Ptolemy writes that the apogee is taken as tropically fixed at $65\tfrac{1}{2}^{\circ}$, and the Sun makes 150 returns in $150^{y}\ 37^{d}$ relative to the apogee. Taken together these of course give the same Hipparchan value $365+\tfrac{1}{4}-\tfrac{1}{300}{}^{d}$ that Ptolemy gave earlier with the simple relations as discussed above. Then later, when the inner planets Mercury and Venus are discussed, Ptolemy says that in $144^{y}\ 37^{d}$ the epicycle makes 144 revolutions plus two sixtieths of one degree relative to the sidereally fixed apogee of each planet, and this is, of course, just the sidereal motion of the Sun.

The ‘simple, unmixed’ mean motions for the Moon were the motion in elongation, η, corresponding to the synodic month, the motion in anomaly, α, and the motion in latitude, ω'. From these and the motion ω_S of the Sun, Ptolemy gives the ‘particular, complex’ motions which correspond to the movement of elements of the lunar model as he describes it. First, he says that the mean motion of the lunar node

$$\omega_N = \omega_S + \eta - \omega'$$

is such that in $37^{y}\ 88^{d}$ the lunar node makes two revolutions in retrograde (clockwise) relative to the zodiac (hence tropical) plus ‘a further one sixtieth of a degree in precise computation’. Second, he says that the motion of the center of the lunar eccentre

$$\omega'_M = 2\eta - \omega'$$

is such that in $17^{y}\ 348^{d}$ a speed equal to twice the elongation minus the argument of latitude makes 203 revolutions in retrograde, minus two sixtieths of a degree.

Third, he says that the motion of the lunar elongation is such that in $19^{y}\ 300^{d}$ the double elongation 2η makes 490 revolutions plus a further four sixtieths of a degree. Finally, he says that in $26^{y}\ 99^{d}$ the anomaly α makes 348 revolutions minus one sixtieth of a degree.

The apogee of each planet, like the sphere of the fixed stars, is taken to move with a motion ω_π of 3° in 300^{ty}, and so is sidereally fixed. Ptolemy defines a ‘particular, complex’ motion ω_P for each planet in terms of ω_π, the solar motion ω_S, and the motion in anomaly ω_a for each planet as follows: for the outer planets

$$\omega_P = \omega_S + \omega_\pi - \omega_a$$

corresponding to the motion of the center of the epicycle on the deferent relative to the apogee of the deferent, while for the inner planets

$$\omega_P = \omega_S + \omega_\pi + \omega_a$$

corresponding to the motion of the planet itself on the epicycle relative to the apogee of the deferent, so ω_P is in both cases a sidereal motion. The variable ω_P for the inner planets is seen nowhere else in Ptolemy's work, but is seen routinely in ancient Hindu astronomy which for many reasons is generally considered to be pre-Ptolemaic.[8]

Thus for Mercury, in $208^y\ 174^d$ the planet on the epicycle makes 865 returns plus four sixtieths of a degree relative to the apogee. For Venus, in $35^y\ 33^d$ there are 57 similar returns plus one sixtieth of a degree. For Mars, in $95^y\ 361^d$ the center of the epicycle makes 51 returns minus three sixtieths of a degree, Jupiter's epicycle in $213^y\ 240^d$ makes 18 returns plus one sixtieth of a degree, and Saturn's in $117^y\ 330^d$ makes 4 returns plus one sixtieth of a degree.

All of these 'complex' period relations are in general agreement with both the corrected and uncorrected *Almagest* relations and the simple *PH* relations discussed above. However, by paying close attention to the tiny remainders that arise 'in precise computation' we can see whether Ptolemy did, as he claims in the introduction, derive them from the simple *PH* relations. While the tiny fractions of a rotation might seem at first glance to be the result of a very delicate calculation, and perhaps even insignificant,[9] that is in fact not the case.

Let us illustrate this using Mercury as an example. The simple period relation for Mercury is 3130 returns in anomaly in 993 sidereal years, which yields a mean motion $\omega_P = 4;5,32,18,25,29,56^{\circ/d}$, and so one sidereal return of the planet on its epicycle takes, using Ptolemy's sidereal year

$$\frac{993}{993+3130}365;15,24,31,32,27,9^d = 87;58,12^d$$

Thus in 88^d Mercury makes one rotation plus the distance it moves in $0;1,48^d$, which is

$$0;1,48^d \cdot 4;5,32^{\circ/d} = 0;7,32^\circ \simeq 7\tfrac{1}{2}' .$$

Ptolemy wants to find a longer period relation such that in an integral number of days Mercury moves an integral number of rotations ± 4′ (we may suppose, since all the excesses he quotes are 4′ or less, but curiously, never zero). He would know that about every

$$\frac{245;32'^{/d}}{7;23'} \simeq 33;15^r$$

returns the excess will once again be small, so he has many revolution numbers to pick from. For example he could pick 133 returns in $32^y\ 20^d$ which has an excess of only 0.1′, or he could pick 599 returns in $144^y\ 134^d$ which has an excess of 4′. What he decided to quote was 865 returns in $208^y\ 174^d$ which has an excess of 3′, but which he somehow miscalculates and gives as 4′. As a point of comparison, if instead of using the mean motion resulting from 4123 returns in 993 sidereal years Ptolemy had used the *Almagest* sidereally-corrected mean motion $4;5,32,18,17,58,38^{\circ/d}$, then 865 returns in $208^y\ 174^d$ would have an excess of only about 0.3′. Therefore we can conclude that for Mercury Ptolemy was using either the simple period relation or something close to it, and not the *Almagest* period relation.

Repeating this calculation for all the complex period relations Ptolemy quotes gives the result shown in Table 3.

Table 3

	PH	Simple *PH*	Uncorrected *Almagest*	Corrected *Almagest*
ω_N	+1	–0.3	–0.3	+0.3
ω'_M	–2	–0.5	–0.2	–0.5
2η	+4	+5.7	+6.5	+6.5
ω_α	–1	–1.3	+7.8	+7.2
Sun	+2	+1.1	+1.1	+1.1
Mercury	+4	+2.9		+0.3
Venus	+1	+0.3		–5.4
Mars	–3	–2.7		+1.0
Jupiter	+1	–0.7		+1.4
Saturn	+1	+2.2		+0.1

The results for the mean motion in anomaly for the Moon confirm that Ptolemy was indeed using the *PH* synodic month and not the *Almagest* month to convert the period relation in anomaly into daily motion. For the lunar node, Jupiter, and Saturn, because the mean motions are so slow, about $3'^{/d}$, $2'^{/d}$ and $5'^{/d}$, respectively, the excess is always 2′ or less for any number of rotations. Also, when Ptolemy says that Saturn makes 4 rotations in $117^y\ 330^d$ with an excess of 1′, he is disguising the fact that using his simple period relation 313 rotations in 324 sidereal years, 4 rotations takes $117^y\ 329^d$ with an excess of only 0.2′, so in 330^d the excess grows to about 2′. In general, there is some scatter in the cases where the excess is only 1′–2′, sometimes to the point of having the wrong sign. For example, the Sun’s excess is in error by nearly 1′, but we know the model parameters he was using (at least according to the text we have), so there is no obvious way to account for a mistake. It does appear, though, that when the excess is larger, the best agreement with the given *PH* values is with the excess values derived from the simple *PH* relations or something approximating them, rather than with the *Almagest* relations.

Altogether, this confirms Ptolemy's statement that his 'particular, complex' relations are derived from his 'simple' relations.

C. PH Epoch Values

In all of his astronomical works Ptolemy generally assumes an epoch date many centuries before his own time, so ultimately his epoch values must be calculated using both measurements of mean longitudes in his own time and relatively precise mean motions. If either or both of those quantities change then we must expect the calculated epoch values to also change. As we have discussed above, except for precession and the Sun, all of the *PH* mean motions have changed from their *Almagest* values, and for that reason alone we can expect the *PH* epoch values to be different from those computed in the *Almagest*. If, in addition, any of the structural parameters in the lunar and planetary models have changed, we would also expect changes in the corresponding mean longitudes in the *PH* not only at the ancient epoch but also at Ptolemy's own time. By and large, this is exactly what we do see.

Just as he did with the *Handy Tables* (hereinafter *HT*),[10] Ptolemy chose noon on Thoth 1 of era Phillip (–323 Nov 12, hereinafter *EP*) as the epoch date of the *PH*. This is 424^y after the *Almagest* epoch date Nabonassar (–746 Feb 26, hereinafter *EN*) and 460^y earlier than the nominal epoch date Antoninus 1 (137 Jul 20, hereinafter *EA*) of the principal *Almagest* observations. Strictly speaking, in the *Almagest* Ptolemy gives an epoch date only for the star catalog, but since all the observations he uses were made within a decade or so of *EA*, we can find the values of all mean longitudes at *EA* using only rough approximations to the mean motions, a circumstance that Ptolemy himself exploits several times in the *Almagest* derivations.

The sphere of the fixed stars and the planetary apogees and nodes, apart from the Sun and Moon, all move with respect to the vernal equinox at the speed of precession, 1° per 100^{ty}, and the *PH* values for all of these in *EP* is exactly what we expect relative to the *Almagest* values. The apogee of the Sun is tropically fixed at $\lambda_A = 65\,{}^1\!/_2^\circ$ and the mean centrum is

$$\kappa = \bar{\lambda}_S - \lambda_A = 162;10^\circ$$

whence

$$\bar{\lambda}_S = \kappa + 65;30^\circ = 227;40^\circ$$

These values follow directly by starting with the *Almagest* value $\kappa = 265;15^\circ$ and adding the increase resulting from 424^y at the daily speed implied by 360° in $365 + {}^1\!/_4 - {}^1\!/_{300}{}^d$. The same epoch value $\kappa = 162;10^\circ$ is also found in the *HT*. If fact, since Ptolemy's model for the Sun is exactly the same in all of his astronomical works: the *Almagest*, the *Canobic Inscription*, the *HT*, and the *PH*, it follows that his solar epoch values must be the same in the *HT* and the *PH*. Conversely, whenever his models differ, especially in the mean motions, we should not expect the same epoch values in the *HT* and the *PH*, even though they have the same epoch date.

The situation for the Moon is more involved, in part because Ptolemy uses different primary reference variables in the *Almagest*, the *HT*, and the *PH*. In the *Almagest* Ptolemy tabulates the mean longitude $\bar{\lambda}$, the mean anomaly α, the argument of latitude ω', and the elongation η. In the *HT* he uses instead the mean anomaly α, the double elongation 2η, the longitude of the northern limit of the deferent $\lambda_N = \omega' - \bar{\lambda}$, and the longitude of the apogee of the deferent $\lambda_A = 2\eta - \bar{\lambda}$. In the *PH* he uses the same primary reference variables as in the *HT* except instead of λ_A he uses the amount by which the double elongation exceeds the argument of latitude

$$\begin{aligned}\omega'_M &= 2\eta - \omega' \\ &= \lambda_A - \lambda_N\end{aligned}$$

Table 4 compares the *PH* epoch values in the Greek and Arabic versions with values computed using the *HT* epoch values and the values resulting from starting with the *Almagest* epoch values at *EN* and adding to the increases due to 424^y at the corrected daily speeds given in the *Almagest*.

Table 4

PH	*PH*(Greek)	*PH*(Arabic)	*HT*	*HT*	*Almagest*	*Almagest*
$\bar{\lambda} = \eta + \bar{\lambda}_S$	178;26°	178;01°	$\bar{\lambda} = 2\eta - \lambda_A$	178;00°	178;00°	$\bar{\lambda}$
α	**85;36°**	**85;17°**	α	**85;17°**	**85;17°**	α
$\omega' = 2\eta - \omega'_M$	48;39°	48;20°	$\omega' = 2\eta - \omega'_M$	48;19°	48;20°	ω'
η	310;46°	310;20°	η	310;20°	310;21°	η
2η	**261;32°**	**260;40°**	2η	**260;40°**	**260;42°**	2η
ω'_M	**(212;53°)**	**212;20°**	$\omega'_M = \lambda_A - \lambda_N$	**212;21°**	**212;23°**	$\omega'_M = 2\eta - \omega'$
λ_N	**230;13°**	**230;19°**	λ_N	**230;19°**	**230;20°**	$\lambda_N = \omega' - \bar{\lambda}$
$\lambda_A = 2\eta - \bar{\lambda}$	83;06°	82;40°	λ_A	82;40°	82;42°	$\lambda_A = 2\eta - \bar{\lambda}$

The value for ω'_M is missing in the Greek text of the *PH*, but it can be reliably reconstructed from the other given values and the mean longitude of the Sun, since $\omega'_M = \eta - \bar{\lambda}_S - \lambda_N$. The values from the *HT* agree within $1'$ with the values that follow from the *Almagest* including the equation of time (for α and ω'_M a $17'$ correction, for 2η a $34'$ correction, and no change for λ_N). The Arabic text of the *PH* agrees well with the *HT* values, but as discussed above, it should not, suggesting that either the Arabic text itself or whatever Greek text it was translated from was at some point altered.

For the planets Ptolemy again uses different primary reference variables in the *Almagest*. the *HT*, and the *PH*. In the *Almagest* Ptolemy gives in Book IX the longitude λ_A of the apogee of the deferent at *EN*, and tables for the mean longitude $\bar{\lambda}$ of the epicycle center and the mean anomaly α of the planet with respect to the apogee of the epicycle. In Book XIII he gives the latitude models, and specifies the orientation of the deferent and epicycle with respect to the apogee of the deferent in terms of two angles: ν is the angle between the northern limit of the deferent and λ_A, while δ is the angle between the northern limit of the epicycle and λ_A. For the outer planets $\delta = \nu + 180^\circ$ and for the inner planets $\delta = \nu - 90^\circ$.

In the *HT* Ptolemy tabulates the longitude of Regulus and gives the (fixed) distance of λ_A from Regulus. Instead of the mean longitude he gives the mean centrum κ which is the distance of the mean longitude from λ_A, so $\bar{\lambda} = \kappa + \lambda_A$, and the usual mean anomaly α. The latitude model, which is slightly different from that found in the *Almagest*, is still specified by giving the angles ν and δ.

In the *PH* Ptolemy first specifies the longitudes of the apogee and northern limit of the deferent, both of which are sidereally fixed, and the mean centrum κ. Instead of using the mean anomaly α, the position of the planet on the epicycle is given in terms of the angle θ from the epicycle apogee to the northern limit of the epicycle, and the angle Ω from the northern limit of the epicycle to the planet. In terms of these variables one has $\nu = \lambda_A - \lambda_N$, $\delta = \theta - \kappa$, $\alpha = \Omega - \theta$, and $\bar{\lambda} = \kappa + \lambda_A$, and inversely $\lambda_N = \lambda_A - \nu$, $\theta = \kappa + \delta$, and $\Omega = \alpha + \theta = \alpha + \kappa + \delta$. In addition, we have as usual for the inner planets $\bar{\lambda} = \bar{\lambda}_S$ and for the outer planets $\bar{\lambda}_S = \bar{\lambda} + \alpha$.

It is straightforward to verify that the planetary epoch values given in the *HT* are all consistent with the epoch values in the *Almagest* up to minor discrepancies of 1′–2′, provided that the increment to the *Almagest* values is computed using the mean motions given in *Almagest* IX 3. Tables 5-9 give the *HT* epoch values, the *PH* epoch values from the Greek and Arabic texts, and the *PH* values that result from a small number of emendations as explained below. In general, since Ptolemy used the same precession constant in all his works, the *PH* and *HT* values for λ_A and λ_N should and do always agree. Since the identical solar model is used in all his works, then for Mercury and Venus $\kappa + \lambda_A$ must equal $\bar{\lambda}_S = 227;40^\circ$, which is the case if we emend κ = 52;16 to 42;16 for Mercury and κ = 177;12 to 177;16 for Venus, and likewise for Mars, Jupiter, and Saturn, $\bar{\lambda} + \alpha = \kappa + \lambda_A + \Omega - \theta$ must also equal $\bar{\lambda}_S = 227;40^\circ$, which is the case if we emend λ_A = 110;44 to 110;54 for Mars, if we accept the Arabic text's $\Omega = 231;16$ for Jupiter, and if we accept all five epoch values for Saturn from the Arabic text, since the entire paragraph containing these is missing in the surviving Greek manuscripts.[11] Unlike the values for the Moon, where the Arabic text agrees with the *HT* instead of the Greek *PH*, for the planets we generally find the Greek and Arabic texts in agreement, and differing from the *HT*, except for the Arabic κ = 356;07 for Mars, which agrees with the *HT* but disagrees with the Greek *PH* value 356;20. The Greek value must be correct since otherwise the relation $\bar{\lambda} + \alpha = \bar{\lambda}_S$ would fail for Mars.

Table 5

Mercury	*PH* (Greek)	*PH* (Arabic)	*PH* (emended)	*HT*	*HT*
λ_A	185;24	185;24	185;24	185;24	λ_A
λ_N	5;24	5;24	5;24	5;24	$\lambda_N = \lambda_A - \nu$
κ	52;16	42;16	42;16	42;16	κ
θ	132;16	132;16	132;16	132;16	$\theta = \kappa + \delta$
Ω	346;41	346;41	346;41	346;56	$\Omega = \alpha + \theta$
$\nu = \lambda_A - \lambda_N$	180	180	180	180	ν
$\delta = \theta - \kappa$	80	90	90	90	δ
$\bar{\lambda} = \kappa + \lambda_A$	237;40	227;40	227;40	227;40	$\bar{\lambda} = \kappa + \lambda_A$
$\alpha = \Omega - \theta$	214;25	214;25	214;25	214;40	α

Table 6

Venus	*PH* (Greek)	*PH* (Arabic)	*PH* (emended)	*HT*	*HT*
λ_A	50;24	50;24	50;24	50;24	λ_A
λ_N	50;24	50;24	50;24	50;24	$\lambda_N = \lambda_A - \nu$
κ	177;12	177;16	177;16	177;17	κ
θ	87;16	87;16	87;16	87;17	$\theta = \kappa + \delta$
Ω	168;35	168;35	168;35	169;18	$\Omega = \alpha + \theta$
$\nu = \lambda_A - \lambda_N$	0	0	0	0	ν
$\delta = \theta - \kappa$	–90;04	–90	–90	–90	δ
$\bar{\lambda} = \kappa + \lambda_A$	227;36	227;40	227;40	227;41	$\bar{\lambda} = \kappa + \lambda_A$
$\alpha = \Omega - \theta$	81;19	81;19	81;19	82;01	α

Table 7

Mars	*PH* (Greek)	*PH* (Arabic)	*PH* (emended)	*HT*	*HT*
λ_A	110;44	110;54	110;54	110;54	λ_A
λ_N	110;44	110;54	110;54	110;54	$\lambda_N = \lambda_A - \nu$
κ	356;20	356;07	356;20	356;07	κ
θ	176;20	176;20	176;20	176;07	$\theta = \kappa + \delta$
Ω	296;46	296;46	296;46	296;46	$\Omega = \alpha + \theta$
$\nu = \lambda_A - \lambda_N$	0	0	0	0	ν
$\delta = \theta - \kappa$	180	180;13	180	180	δ
$\bar{\lambda} = \kappa + \lambda_A$	107;04	107;14	107;04	107;01	$\bar{\lambda} = \kappa + \lambda_A$
$\alpha = \Omega - \theta$	120;26	120;26	120;26	120;39	α
$\bar{\lambda}_S = \bar{\lambda} + \alpha$	227;30	227;27	227;40	227;40	$\bar{\lambda}_S = \bar{\lambda} + \alpha$

Table 8

Jupiter	*PH* (Greek)	*PH* (Arabic)	*PH* (emended)	*HT*	*HT*
λ_A	156;24	156;24	156;24	156;24	λ_A
λ_N	176;24	176;24	176;24	176;24	$\lambda_N = \lambda_A - \nu$
κ	292;43	292;23	292;43	292;20	κ
θ	92;43	92;43	92;43	92;20	$\theta = \kappa + \delta$
Ω	NA	231;16	231;16	231;17	$\Omega = \alpha + \theta$
$\nu = \lambda_A - \lambda_N$	–20	–20	–20	–20	ν
$\delta = \theta - \kappa$	160	160	160	160	δ
$\bar{\lambda} = \kappa + \lambda_A$	89;07	89;07	89;07	88;44	$\bar{\lambda} = \kappa + \lambda_A$
$\alpha = \Omega - \theta$	NA	138;33	138;33	138;57	α
$\bar{\lambda}_S = \bar{\lambda} + \alpha$	NA	227;40	227;40	227;40	$\bar{\lambda}_S = \bar{\lambda} + \alpha$

Table 9

Saturn	*PH* (Greek)	*PH* (Arabic)	*PH* (emended)	*HT*	*HT*
λ_A	NA	228;24	228;24	228;24	λ_A
λ_N	NA	188;24	188;24	188;24	$\lambda_N = \lambda_A - \nu$
κ	NA	210;38	210;38	211;02	κ
θ	NA	70;38	70;38	71;02	$\theta = \kappa + \delta$
Ω	NA	219;16	219;16	219;18	$\Omega = \alpha + \theta$
$\nu = \lambda_A - \lambda_N$	NA	40	40	40	ν
$\delta = \theta - \kappa$	NA	220	220	220	δ
$\bar{\lambda} = \kappa + \lambda_A$	NA	79;02	79;02	79;26	$\bar{\lambda} = \kappa + \lambda_A$
$\alpha = \Omega - \theta$	NA	148;38	148;38	148;16	α
$\bar{\lambda}_S = \bar{\lambda} + \alpha$	NA	227;40	227;40	227;42	$\bar{\lambda}_S = \bar{\lambda} + \alpha$

Ptolemy concludes Book 2 of the *Planetary Hypotheses* by describing a set of tables for the time dependence of various mean positions associated with each of his planetary models. The tables have not survived but the text defines the variables he proposes to tabulate as follows:

for the Sun:
1) κ, the Sun from the apogee of the deferent in the trailing (clockwise) direction

for the Moon:
1) λ_N, the northern limit of the deferent from the vernal equinoctial point in the leading (counterclockwise) direction
2) ω'_M, the apogee of the deferent from the northern limit in the leading direction
3) 2η, the centre of the epicycle from the apogee of the deferent in the trailing direction
4) α, the centre of the moon from the apogee of the epicycle in the leading direction

for the five planets:
1) λ_A, the apogee of the deferent from the vernal equinoctial point the trailing direction
2) κ, the centre of the epicycle from the apogee of the deferent in the trailing direction
2a (for Mercury only) the centre of the deferent from the apogee in the leading direction
3) θ, the northern limit of the epicycle from the apogee of the epicycle in the leading direction
4) Ω, the planet from the northern limit of the epicycle in the trailing direction

As it happens there is no discussion of any of these variables elsewhere in Book 2, but in the first part of Book 1 (the only part surviving in Greek), following his introduction, Ptolemy has written two paragraphs for each planet, the first describing his model and giving the mean motion of each angle that he later says comprise the tables, the second giving the *EP* epoch value of each of the angles. In the text of Book 1 Ptolemy also gives for the planets λ_N, the northern limit of the eccentre from the vernal equinoctial point towards the trailing parts of the cosmos, but this is not in his tables, presumably since there is always a constant angle $\nu = \lambda_A - \lambda_N$ between the apogee and the northern limit of the eccentre.

The fact that Ptolemy clearly gives us both the mean motions and the epoch values of the variables he is tabulating would allow us to reconstruct the lost tables with some confidence, but that would not teach us much. What is much more instructive is to compute the value of each variable in Ptolemy's time, at say epoch *EA*, and to compare those with the corresponding values computed using the *Almagest*. Such comparisons are of interest for two reasons:

(1) any differences in the *Almagest* and *PH* values at *EA* will most likely come from changes in the contemporary observations that determine the model structural parameters: eccentricity *e*, epicycle radius *r*, and various mean longitudes, while differences in the model parameters at distant epochs such as *EP* are affected also by the changes in the mean motions, and
(2) from the pattern of changes in the model parameters we might learn something about the analysis procedure that produced those changes.

The mean values to consider for the Moon are: 2η, ω'_M, λ_N, and α, from which the mean longitude $\bar{\lambda}$ and the argument of latitude ω' can be computed. For the outer planets the relevant variable is the mean centrum κ and for the inner planets the distance Ω of the planet from the northern limit of the epicycle, or alternatively the mean anomaly $\alpha = \Omega - \kappa - \delta$. All other values should agree due to the use of the same precession constant and solar motion in the *PH* and the *Almagest*, and they would agree if not for the error of 1′ in the period relation for κ for the inner planets (as mentioned above, the text gives $144^r + 2'$ in $144^y\ 37^d$, but it should give 1′). Over the 460^y between *EP* and *EA* this gives an error of about 3′ in the mean longitude of Mercury and Venus, and to keep things as clean as possible that error has been corrected in the comparisons that follow. The values of interest are shown in Tables 10 and 11.

Table 10

Moon	*PH*(*EP*)	*Alm(EP)*	diff.	*PH*(*EA*)	*Alm(EA)*	diff.
2η	261;32°	260;42°	0;50°	353;38°	353;46°	–0;08°
ω'_M	212;53	212;23	0;30	297;07	297;16	–0;09
λ_N	230;13	230;20	–0;07	123;49	123;47	0;02
α	85;36	85;17	0;10	213;57	216;02	–2;23
$\bar{\lambda} = \eta + \bar{\lambda}_S$	178;26	178;00	0;26	292;41	292;42	–0;01
$\omega' = 2\eta - \omega'_M$	48;39	48;20	0;19	56;30	56;29	0;01

Table 11

	PH(*EP*)	*Alm(EP)*	Diff.	*PH*(*EA*)	*Alm(EA)*	Diff.
Ω (Mercury)	346;41°	346;55°	–0;14°	205;09°	205;15°	–0;06°
Ω (Venus)	168;35	169;16	–0;41	245;22	244;39	0;43
α(Mercury)	214;25	214;40	–0;15	274;32	189;26	–0;06°
α(Venus)	81;19	82;00	–0;41	189;20	273;49	0;43
κ (Mars)	356;20	356;07	0;13	141;08	141;08	0;00
κ (Jupiter)	292;43	292;19	0;24	204;25	203;59	0;26
κ (Saturn)	210;38	211;00	–0;22	68;49	69;07	–0;18

The *PH* and *Almagest* mean values for longitude and latitude of the Moon are, allowing for rounding errors, in agreement at *EA*, but the mean value of anomaly shows a relatively large change. Now in the *Almagest* an elegant geometrical analysis of a trio of lunar eclipses takes as empirical input the times of the three eclipses and delivers as output the lunar eccentricity (Ptolemy gets $r / R \simeq 5;15 / 60$), the mean longitude $\bar{\lambda}$, and the mean anomaly α. It happens, however, that the values derived in such analyses are very sensitive to changes in the input times, a circumstance that Ptolemy himself explains in detail in *Almagest* IV 11, and the values are strongly coupled, so that in general they all change. This means that the pattern of changes we see in the *PH* – a change in α but no change in *r/R* or $\bar{\lambda}$ – is quite surprising, and probably means that Ptolemy was making changes to the value of mean anomaly based on empirical considerations other than eclipse trios. Furthermore, it bears repeating that we do not know what lunar parameter values Ptolemy was in fact changing to get to the *PH* values.

We find a very similar pattern in the case of Saturn and Jupiter. In both cases the *Almagest* analysis is an even more elaborate geometrical exercise, this time of trios of oppositions with the mean Sun.[12] Once again we know that the results of the analysis – in this case the eccentricity *e/R*, the longitude of the apogee λ_A, and the mean centrum κ – are sensitive to small changes in the input data, and when one changes, they all change.

Yet when we compare the *Almagest* and *PH* parameter values, for both planets κ changes but λ_A is unchanged, while for Jupiter *e/R* is unchanged and for Saturn it is either unchanged (as in the Arabic text) or the change is very small (as in the Greek text, which reads $3\frac{1}{3}$ instead of the *Almagest*'s $3\frac{1}{3}\frac{1}{12}$). In addition, in the *Almagest* the values for *r/R* are determined in a subsequent analysis based on an additional observation for each planet, and the result for *r/R* depends on the values assumed for *e/R,* λ_A, and κ, yet in both cases the *Almagest* and *PH* values for *r/R* are in agreement, even though κ has changed. Once again this strongly suggests that Ptolemy was basing his change in κ for Saturn and Jupiter on an analysis different in character from what we find in the *Almagest*. On the other hand, for Mars at epoch *EA* the values of *e/R,* λ_A, κ, and *r/R* are the same in the *Almagest* and the *PH*, so it appears that except for the mean motion in anomaly Ptolemy found no reason to change the parameters for Mars.

For the inner planets the analyses in the *Almagest* are much simpler than we find for the outer planets. Ptolemy shows how to use successive pairs of observations at greatest elongation to find in turn λ_A, e_1, e_2, and *r/R*, and a final observation gives the mean anomaly α. The change of 6′ in α (or Ω) for Mercury might be due in part to the fact that in the *Almagest* and the *HT* $e_1 = e_2 = 3$ and r = 22½, while in the *PH* $e_1 = 3$ but $e_2 = 2\frac{1}{2}$ and r = 22 ¼. For Venus the model structural parameters are the same in all of Ptolemy's works ($e_1 = e_2 = 1\frac{1}{4}$ and $r = 43\frac{1}{6}$), so the *Almagest–PH* change of 43′ in α cannot be explained by a structural parameter change.

Summary

Early in his introduction Ptolemy acknowledges "the corrections that we have made in many places", and indeed, apart from the sphere of the fixed stars and the Sun, he has changed many things. He first gives eight long period relations for the mean motions – longitude, anomaly, and latitude for the Moon, and anomaly for each of the five planets – which he says are "approximations to the restitutions computed from the correction", and although he does not tell us specifically what those restitutions or the corrections are, it seems clear that all of them, with the possible exception of Mercury and Saturn, are not what we know from the *Almagest*.

He next gives 'particular, complex' periods which are approximations to linear combinations of the eight approximations and the periods for the fixed stars and the Sun, and he also gives the values of the corresponding mean positions associated with each of these period relations. Altogether, these then provide the basis for the lost tables that he does not mention until the final sentences of the entire work.

All of the epoch values for the Moon and the planets are different at *EP*, but this is in part a consequence of the changes in the mean motions. In Ptolemy's time at epoch *EA* the changes are to the anomaly, but not the longitude or latitude, of the Moon, the mean longitude of Saturn and Jupiter, but not Mars, and the anomaly of Venus and Mercury, the former a large change, the latter a small one. Finally, the pattern of parameter changes we see suggests that the analyses that yielded the *PH* parameters were not the elegant trio analyses of the *Almagest* but some sort of serial determinations of the parameters based on sequences of independent observations.[13]

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[2] G. J. Toomer, *Ptolemy's Almagest* (1984).
[3] O. Neugebauer, *A history of ancient mathematical astronomy*, (1975), 781-833 is the only substantial work on the Greek part of the *PH*.
[4] Hamilton, N. T., N. M. Swerdlow, and G. J. Toomer, "The Canobic Inscription: Ptolemy's Earliest Work." In J. L. Berggren and B. R. Goldstein, eds., *From Ancient Omens to Statistical Mechanics.* Copenhagen (1987) 55–73.
[5] D. H. Fowler, *The Mathematics of Plato's Academy*, (1987) has a nice discussion of approximation by a succession of ratios, and B. R. Goldstein, "On the Babylonian Discovery of the Periods of Lunar Motion", *Journal for the History of Astronomy*, **33** (2002) 1-13 illustrates the use of such approximation schemes in another context.
[6] That he was using the *PH* month and not the *Almagest* month will be confirmed when his 'particular, complex' approximations are discussed in the next section.
[7] Alexander Jones and Dennis Duke, "Ptolemy's Planetary Mean Motions Revisited", *Centaurus,* 47 (2005) 226-235.
[8] Dennis W. Duke, "Mean Motions and Longitudes in Indian Astronomy, *Archive for History of Exact Sciences*, **62** (2008) 489-509.
[9] O. Neugebauer, *op. cit.* (ref. 3) 903.
[10] Heiberg, *op. cit.* (ref. 1) has the Introduction, W. D. Stahlman, "The Astronomical Tables of Codex Vaticanus Graecus 1291", Ph.D. dissertation, Brown University, 1960, is an English translation of and commentary on the tables.
[11] Heiberg, *op. cit.* (ref. 1) used two Greek manuscripts, A and B. MS B ends just before the paragraph which should have the epoch values would begin, while MS A continues to the end of the Saturn section but leaves blank spaces where the epoch values would go. Heiberg inserted numbers from the Arabic manuscripts into those spaces. Jones, *op. cit.* (ref. 1) suggests that the Saturn text for A was a restoration made by copying from the Jupiter text.
[12] Dennis W. Duke, "Ptolemy's Treatment of the Outer Planets", *Archive for History of Exact Sciences*, **59** (2005) 169-187.
[13] J. Evans, *The Theory and Practice of Ancient Astronomy*, (1998), 362-368 works through an example of this for Mars.