# The Ancient Values of the Planetary Parameters of Venus and Mercury

Dennis W Duke, Florida State University

In chapters IX-XI of the *Almagest*[1] Ptolemy produces a rather small set of observations for each of the five planets, most of which he specifically claims to have made himself, and proceeds to systematically use those observations to derive each of the parameters of his final planetary models: a rather complicated crank mechanism for Mercury, and the equant model for Venus, Mars, Jupiter and Saturn. Wilson, Newton, and Swerdlow have thoroughly analyzed Ptolemy's presentations.[2] Each concludes that Ptolemy simply did not do what he wrote that he did in his *Almagest* presentations on Venus and Mercury, and Newton claimed the same for the outer planets Mars, Jupiter and Saturn. Instead, they show that Ptolemy very likely already knew the values of the parameters of his model and adjusted his 'observations' to make his 'derivations' of those parameters appear direct and simple.

So if Ptolemy inherited the values of the parameters, or if he derived them himself from some prior analysis that he chose not to leave us, then the question is: how would one use ancient data to derive fairly accurate values for these parameters? As many previous commentators have assumed, the most plausible scenario is that the ancients had somehow managed to assemble a fairly substantial set of observations, perhaps over a fairly long interval of years. So one way to try and understand the question is to assemble for ourselves a set of observations that could plausibly have been available to an ancient astronomer, and then try and analyze those observations in the context of Greek geometrical models to see what parameters emerge.

The principal observations used for the inner planets are greatest elongations. Ptolemy defined the elongation of an inner planet as the difference in longitude of the planet and the mean Sun, and we shall assume that his predecessors did likewise. Since the longitude of the mean Sun is obtainable only in the context of a theory of the Sun's motion, we know that insofar as elongations are used to fix the parameters of planetary models, the existence of a reasonably good model of the Sun is then a prerequisite. Now measurements of planetary longitudes were generally made relative to the longitudes of reference stars, and the tropical longitudes of reference stars must be made with respect to the Sun (and need not coincide with the observation of the planet), so any error in the solar model will be directly transmitted as an error in the planet's tropical longitude. Thus the elongations will be somewhat immune to the simplest errors in the solar theory, such as a misplaced equinox. If, however, the errors in the solar theory grow with time (as, e.g. in Hipparchus' solar theory), then pairs of morning and evening elongations at the same longitude of the mean Sun will incur errors, and these errors will be most apparent when we compute the difference of the absolute values of the elongations.

In order to generate samples of historical data I use the planetary models of Bretagnon and Simon,[3] which yield geocentric longitudes and latitudes for the Sun and the planets as far back as 4000 BC, and to far greater accuracy than needed for this investigation. Figures 1 and 2 show the evening (i.e. positive) and morning (i.e. negative) elongations of Venus as a function of the Sun's mean longitude using positions computed at five day intervals over 400 BC – 150 BC. The outer envelopes of values thus determine the morning and evening greatest elongations  as a function of solar mean longitude.[4] Figure 3 shows the sum of the absolute values of greatest morning and evening elongations as a function of solar mean longitude, while Figure 4 shows the algebraic sum of the greatest evening and morning elongations, i.e. the difference of their magnitudes. Figures 5-8 show the corresponding results for Mercury. For comparison and later reference, the figures also include the corresponding result from using the *Almagest* models of Venus, Mercury and the mean Sun to generate planetary positions. And although I am using charts throughout this paper, the ancient analyst was most likely using tables of numbers. We do know, though, that the analysts were very proficient at using tables. They could not only interpolate, but also find local maxima, minima, and rates of change of their tabulated functions. Presumably all of the analysis I do with graphs below was done just as well with tables in ancient times.

The set of data so collected is clearly of far greater quality than we can reasonably expect for an ancient data collection, and the issues associated with that point will be addressed below. But for now let us assume that such a data collection is available and see how it might be analyzed in the context of Greek geometrical models. This will show us at least what is ideally possible, and hence provide an initial frame of reference for the later analysis of more realistic sets of observations.

Of the possible Greek geometrical models we shall consider three. First is the simple model known at least as far back as Apollonius: a concentric deferent of radius $R$ with an epicycle of radius $r$. Second is an intermediate model which the *Almagest* refers to only indirectly,[5] but which we can be fairly sure was at least considered at some point: an eccentric deferent with radius $R$ and eccentricity $e$, and an epicycle of radius $r$. And third is the *Almagest* model (excepting Mercury): an eccentric deferent with radius $R$ and eccentricity $e$, and an epicycle of radius $r$ which moves uniformly about the equant, a point which lies on the apsidal line a distance $e'$ from the Earth. In all three models the planet revolves uniformly around the epicycle with a period in anomaly, and the center of the epicycle revolves around the deferent with the period of the mean Sun. In the first two models the center of uniform motion is the center of the deferent, while in the third model the center of uniform motion is the equant.

The parameters of the models are therefore: (1) the mean motions in longitude and in anomaly, (2) the direction of the apsidal line, and its change in direction with time, (3) the radius $r$ of the epicycle, (4) the eccentricity $e$ of the deferent, (5) the distance $e'$ between the Earth and the center of uniform motion (the equant), and (6) the values of mean longitude, anomaly, and apogee at some initial time. In addition, for the inner planets Ptolemy makes the assumption that the direction of the line from the center of uniform motion to the epicycle center is parallel to the line from the Earth to the mean Sun. We do not know whether astronomers earlier than Ptolemy assumed this, but in the following we shall assume that they did.

Now for all five planets, various period relations were very well known and clearly could have been used to derive the mean motions. In addition, a single observation of longitude at a known time $t$ is sufficient to fix the initial values once the other parameters are decided. So for all five planets the principal problem is to find values for the direction of the apsidal line, the epicycle radius $r$, the deferent eccentricity $e$, and the distance $e'$ of the equant from the Earth.

For Venus and Mercury, the most obvious quality we notice is that the greatest elongations are not constant as the planet traverses the zodiac. Presumably this was realized very early, and so the ancient astronomers would have known that the simplest Apollonius model with a simple epicycle on a concentric deferent could not work. Now for a given distance $R$ between observer and epicycle center, and a given radius $r$ of the epicycle, the greatest elongation $\eta$ results when the line of sight from Earth to the planet is tangent to the epicycle, so that $r$ is determined by the simple relation $r = R\sin\eta$. Since $\eta$ is observed to be not constant, then either the epicycle radius $r$ or the distance $R$ to the epicycle center, or both, must be varying. The ancient Greek analysts apparently always chose to keep the epicycle radius $r$ fixed. It is quite plausible, however, that they realized that they could estimate the epicycle radius by simply observing the *average* greatest elongation, which will occur when the epicycle center is its average distance from the observer. Using the conventional norm $R = 60$ and the data shown in Figures 1 and 2, which yield an average elongation of about 46.2°, the implied epicycle radius is 43;20. When rounded, this agrees exactly with the value attributed by Pliny to Timaeus of 46°, which implies an epicycle radius of 43;10, the value Ptolemy uses in the *Almagest*. The average value of the elongations for Mercury, shown in Figures 5 and 6, is about 22.3°, which also rounds to the value of 22° that Pliny attributes to Cedenas and Sosigenes, and which leads to $r = 22;28,35 \approx 22;30$, and again the value Ptolemy uses. Thus the epicycle radii for the inner planets follow simply from knowledge of the average greatest elongations, and since they were apparently known long before Ptolemy's time, we might have some confidence that enough greatest elongations were observed to provide adequate estimates of the average.

The next task is to determine the direction of the apsidal line. One idea is that apogee is the direction in which the sum of greatest evening and morning (absolute) elongations is minimum, and hence the epicycle is farthest from the observer, while perigee is the direction in which the sum is largest. This method is also the least sensitive to any error in the computed position of the mean Sun. Ptolemy alludes to this method in *Almagest* X.2 when he says "Furthermore, it has also become plain to us that the eccentre of Venus carrying the epicycle is fixed, since nowhere on the ecliptic do we find the sum of the greatest elongations from the mean [Sun] on both sides to be less than the sum of both in Taurus, or less than the sum of both in Scorpius." Thus Figure 3 shows that apogee for Venus is around 52° and that perigee is about 180° away, around 232°. It might have also occurred to the ancient analyst to ask for the direction in which the morning and evening elongations sum (algebraically) to zero. Figure 4 shows that this occurs at about 60° and

236° for the real data. The positions for apogee and perigee that result from these two methods are not equal due to various technical reasons: the true orbits are elliptical, the orbits do not lie in a common plane, and the true mean Sun does not lie on the line between the center of Earth's orbit and the center of Venus' orbit.[6] It is also clear from Figures 3 and 4 that the symmetry method that Ptolemy claimed to be using to find the direction of the apsidal line should have produced a result much closer to 60° than to 52°, whereas he in fact produced a result, adjusting for his movement of the apsidal line, close to 52°. This suggests that he was using a symmetry method with data adjusted to give a result that he had inherited, and that result was fairly accurately derived from more accurate data using the sums method.

Consideration of the sums of greatest elongations for Mercury (see Figure 7) shows that the direction of the apogee for the real data is at about 220° and perigee is about 180° away at 40°. On the other hand, the difference method (see Figure 8) gives an apogee for Mercury at about 205° and a perigee at about 28°. The *Almagest* model data give an apogee at just over 180° and, by construction, double perigees about 120° away from apogee.

Finally, we look at the determination of the eccentricity and the position of the equant. Let's begin by thinking in terms of the simplest model that might work, the intermediate model which has an eccentric-deferent and an epicycle. In this model the center of the deferent and the center of uniform motion are the same, and so $e = e'$. The catch, however, is that there is one way to estimate $e$ and another way to estimate $e'$, and as we shall shortly see, these two different methods give different estimates.

First we estimate $e'$. If the mean Sun is at longitude $\lambda_{\overline{S}}$ and the longitude of the apogee is at $\lambda_A$, then the equant distance $e'$ is given by

$$\frac{e'}{R} = \frac{\sin c}{\sin(\alpha - c)}$$

where $c = (\eta_M - \eta_E)/2$ and $\alpha = \lambda_{\bar{S}} - \lambda_A$. This method is only useful, however, when the mean Sun is well away from apogee or perigee, and when the two elongations are close enough in time that the apogees of the observations are not significantly different. If we analyze the real data using such a model we can estimate *e*′ when the longitude of the mean Sun is near quadrant. The resulting estimated value of *e*′ is about 1.85, assuming $R = 60$.

On the other hand, the ancient analyst would estimate *e*, the eccentricity of the deferent, using elongations as close as possible to the apsidal line. In fact, if $\eta_P$ and $\eta_A$ are elongations at perigee and apogee, respectively (and evening or morning doesn't matter by symmetry, so one could also just average the morning and evening elongations near apogee and perigee), then the eccentricity *e* is given by

$$\frac{e}{R} = \frac{\sin\eta_P - \sin\eta_A}{\sin\eta_P + \sin\eta_A}$$

Near the apogee of the real data the eccentricity *e* is about 0.9 (assuming *R* =60). What is particularly clear is that the ratio *e*′/*e* is close to 2 for the real data. Therefore these two relatively simple analyses send a clear signal that for Venus the center of the deferent is closer to the Earth than the center of uniform motion, in contradiction to the assumption of the intermediate model. It is plausible, then, that it was the need to reconcile this contradiction that led to the creation of the equant model that we find for Venus in the *Almagest*. It is certainly the case that for Venus, and only for Venus, Ptolemy presents an analysis that closely parallels the above to explain the problem that needs to be resolved.

Repeating the equant determination using Mercury data leads to $e' = 3.25$ and $e = 4.6$, implying that the center of uniform motion is closer to the Earth than is the center of the deferent, and thus opposite to the situation with Venus. This is an unavoidable consequence of the fact that the center of Mercury's orbit is farther from the Sun than the center of Earth's orbit.[7] We can only surmise that perhaps the difficulty of reconciling the different sizes of the eccentricity and the equant led to the creation of a special model for Mercury. Perhaps it is also possible that Ptolemy (or whoever invented the model) could not find a way to make the crank mechanism account for the variation in greatest elongation without also introducing a double perigee, but in absence of more detailed analysis we can say very little with certainty.

Each of these determinations of the equant and the eccentricity involves getting a small number from the difference of two experimentally measured larger numbers, and hence all the estimates unavoidably have substantial relative errors. Thus it is not surprising that the values used in the *Almagest*, whoever they might originate from, differ at the 15-20% level from our more exact estimates which use accurate data.

Overall then we see that given an adequate (and some might say extravagant) base of historical data, it is plausible that straightforward analysis using techniques that we expect were accessible to ancient astronomers leads to just the results for Venus and Mercury that we find in the *Almagest*. However, all the above is based on accurate values of longitudes sampled at 5 day intervals over a period of 250 years. In reality, of course, the measurements would not have been so accurate, nor the sampling nearly so regular, nor the interval necessarily so long. In addition, especially for Mercury, one should account for the fact that the planets are not always visible.

For Venus the primary issue is the length of the time interval over which observations were available. To take an extreme example, let us suppose that Ptolemy only had data over the time period explicitly mentioned in the *Almagest*, 127-141 AD. The available data[8] are shown in Figures 9 and 10, and reflect the periodicity of Venus' orbit. Estimating the direction of the line of apsides from Figure 10 alone would be hazardous at best, and even if you somehow knew the direction of apogee there are not enough observations near apogee, perigee or quadrant to make a good estimate of the eccentricity or the equant lengths. In fact, unless one includes observations prior to about 50 AD one could not know that the maximum sum of elongations was actually in Scorpio, as Ptolemy tells us in *Almagest* X.2.[9]

How many years of Venus observations are enough to get useful results? Figure 11 shows the results of collecting data for 100 years. Since the general trend of the curve is at least partially defined now at a number of points, one can get an impression of the longitudes of the maximum and minimum sums, and by interpolation estimate any needed values. So it appears that if the ancient analyst had access to about a century's worth of data, he would be able to use that data as we have discussed to estimate the needed parameters in the intermediate model. More data might have been available, but the fact is we have no direct evidence that such data series ever existed. What we are showing in this paper is that *if* the data existed, *then* it is plausible that the ancient analyst could use the data to estimate values of the parameters of geometrical models.

For Mercury the primary issue is not so much the length of the time interval of the observations as the difficulty of observing Mercury at different times of the year. To once again take an extreme example, we use the time interval 127-141 AD. In order to make sure we record Mercury only when it is visible, we now generate longitudes every 6 minutes. However, we also compute the altitudes of the Sun and Mercury assuming we are at Alexandria, and we record the observation only if the Sun is 5° or more below the horizon and if Mercury is 5° or more above the horizon. The results are shown in Figures 12 and 13. It is sometimes said[10] that the shallow angle of the ecliptic during Spring mornings and Fall evenings would make the observations of Mercury difficult if not impossible. In the context of Ptolemy's *Almagest* analysis of Mercury, there are two needed observations that are conspicuously missing: the morning of 131 Apr 4 and the evening of 138 Oct 4. Precise calculation shows, however, that according to the visibility conditions being used here,[11] Mercury was visible on the first date for about 16 minutes and on the second date for about 8 minutes. And although these intervals of opportunity are narrow, they existed for about a week on both sides of the target date. However, even allowing for considerable further degradation of the data, and omitting the Spring morning and Fall evening elongations, it is clear that adequate data might well have been available to allow a determination of model parameters as discussed above. Furthermore, the idea that the difficulty in observing Mercury would lead to relatively fewer observations of Mercury is not supported by the historical records, since, as pointed out by Swerdlow,[12] the *Astronomical Diaries*[13] contain nearly three times as many observations of Mercury as of Jupiter and Saturn, and LBAT 1377,[14] a text devoted to Mercury, contains more observations than all the surviving *Diaries*.

For the outer planets the observation of choice is the opposition, at which the planet, the Earth and the mean Sun are aligned (with the Earth in the middle). In the *Almagest* Ptolemy uses an elegant geometrical analysis using three oppositions to determine the direction of the apsidal line and the size of the equant. Evans, however, has suggested a much simpler method that uses a time history of oppositions to accomplish the same goal.[15] To locate the apogee of the deferent, one considers the average distance between oppositions as a function of the longitude of the oppositions. This function is minimum at the longitude of the apogee. Examples for Mars, Jupiter and Saturn are shown in Figure

14 using oppositions that occurred over the (arbitrary) interval 250 BC – 150 BC. Then by picking an opposition as near as possible to apogee and a second opposition at some other longitude, one can compute in the context of the intermediate model the effective size of the equant (or eccentricity, they are equal in the intermediate model) as seen from different longitudes of the (second) opposition using the formula

$$e / R = \sin c / \sin a,$$

where $a = \lambda - \lambda_0$, $\alpha = \omega_t (t - t_0)$, and $c = \alpha - a$. In these equations $\omega_t$ is the mean motion in longitude, the opposition at apogee has longitude $\lambda_0$ at time $t_0$, while the second opposition has longitude $\lambda$ at time $t$. Assuming that the distance $R = 60$ is constant, one finds for all three outer planets an effective equant distance that decreases as you move away from apogee, as shown in Figure 15. Or inversely, if one chooses to keep the effective equant distance constant, then it must be that the distance $R$ is *increasing* as the planet moves away from apogee. This is, of course, precisely what happens to the distance between the center of uniform motion and the center of the epicycle in the *Almagest* equant model.

Evans has also shown another approach to motivating the *Almagest* equant model that uses the varying width of opposition loops and their unequal spacing in longitude.[16] It is certainly possible that either method, or perhaps both, provided the motivating factors that first exposed the inadequacy of the intermediate model, and then suggested a solution. In any event, though, the fact that the anciently attested values for all five planets agree so well with the results from modern calculation shows that the ancient observations, however they were collected and analyzed, must have been adequate for the purpose.

In summary, the various analyses show simple and accessible methods whereby ancient astronomers might well have used time histories of the longitudes of planets, combined with a solar model and the longitudes of a few bright stars near the ecliptic, to estimate the parameters of their models. The primary technical supplement to the geometrical models might well have been extensive sets of tables, just as Ptolemy himself eventually uses in the *Almagest*. Coupled with the nature of Ptolemy's own presentations in the *Almagest*, as discussed by Wilson, Newton, and Swerdlow, these results therefore suggest that such practice predates the analyses Ptolemy left us in the *Almagest*.

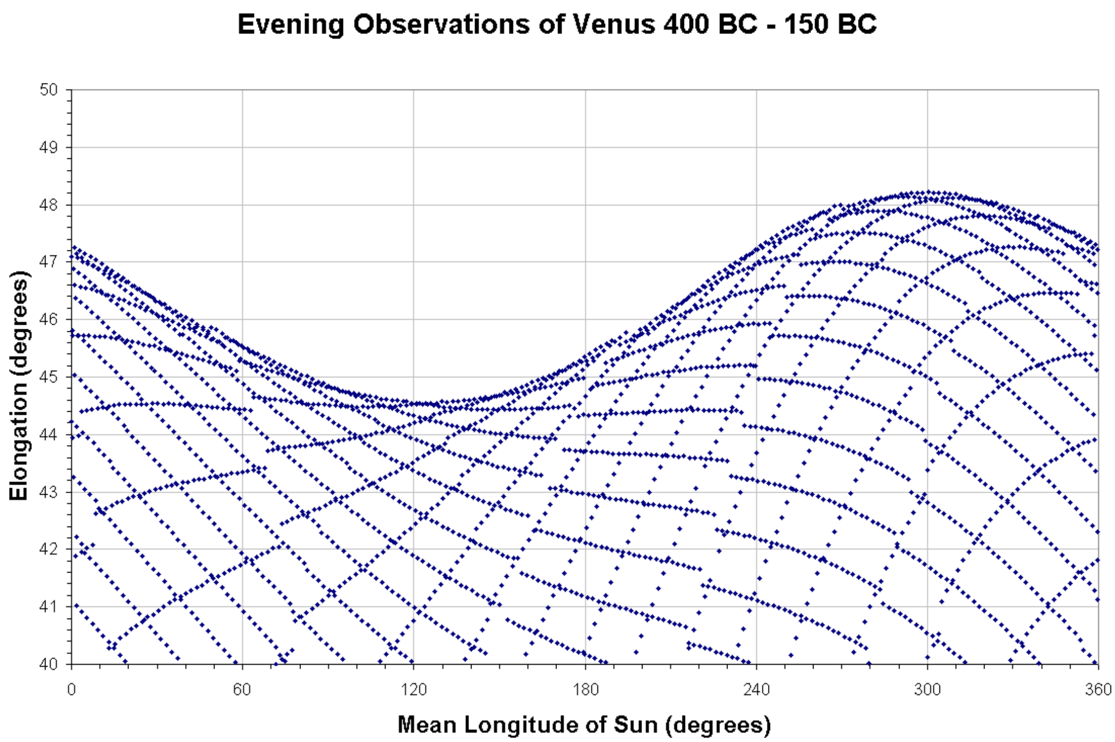



Figure 1.

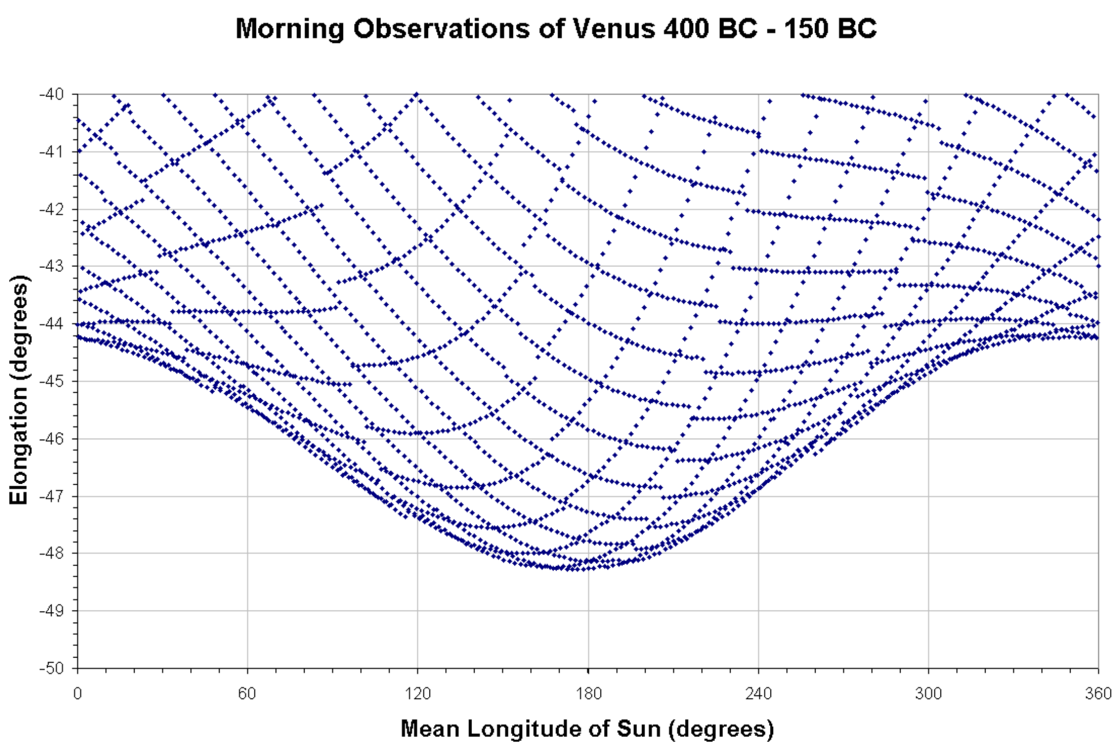



Figure 2.

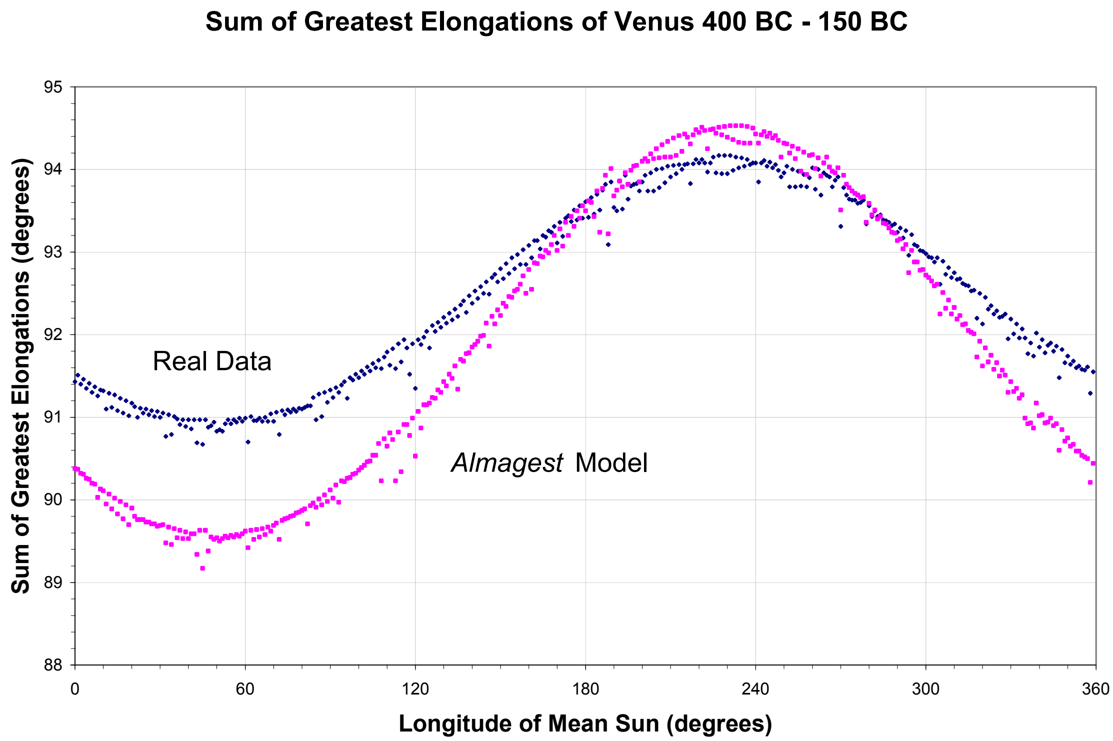



Figure 3.

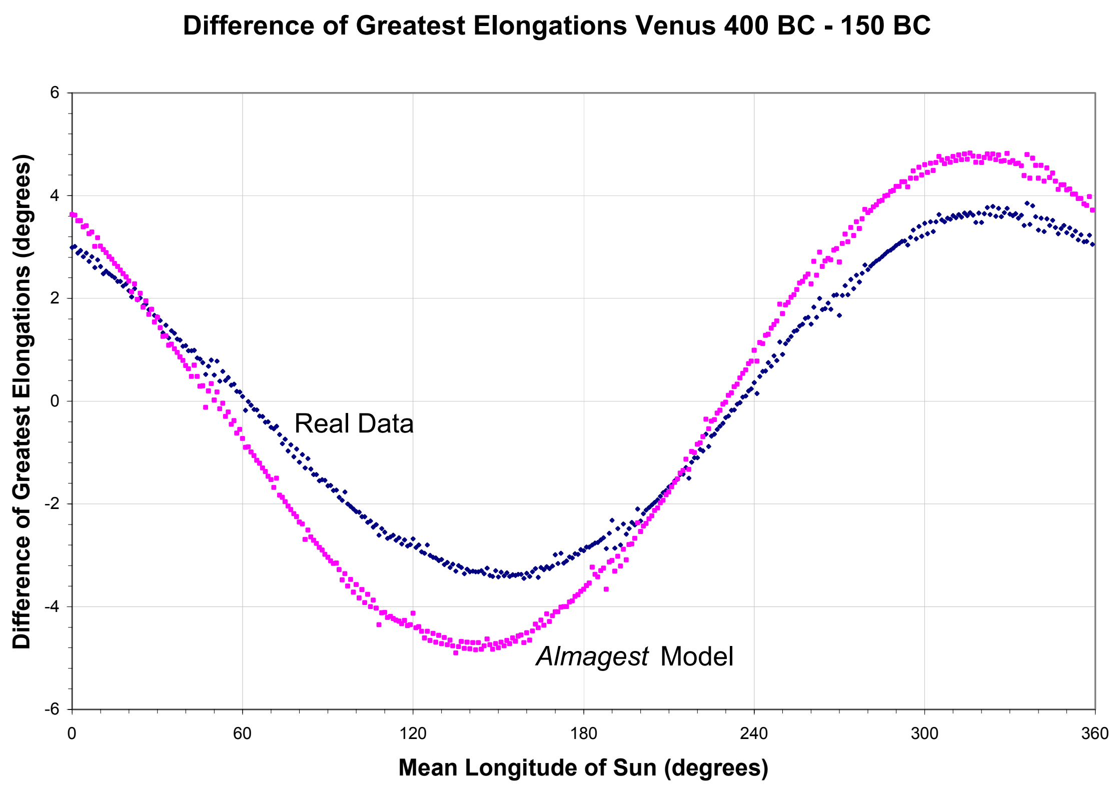



Figure 4.

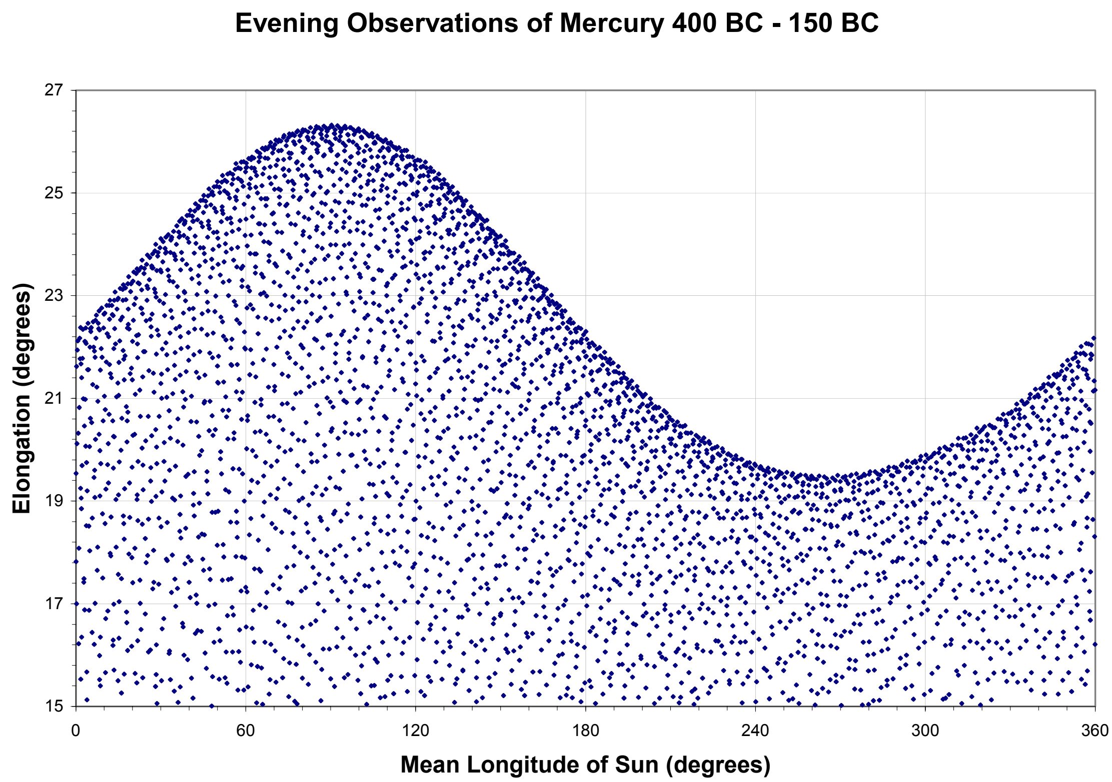



Figure 5.

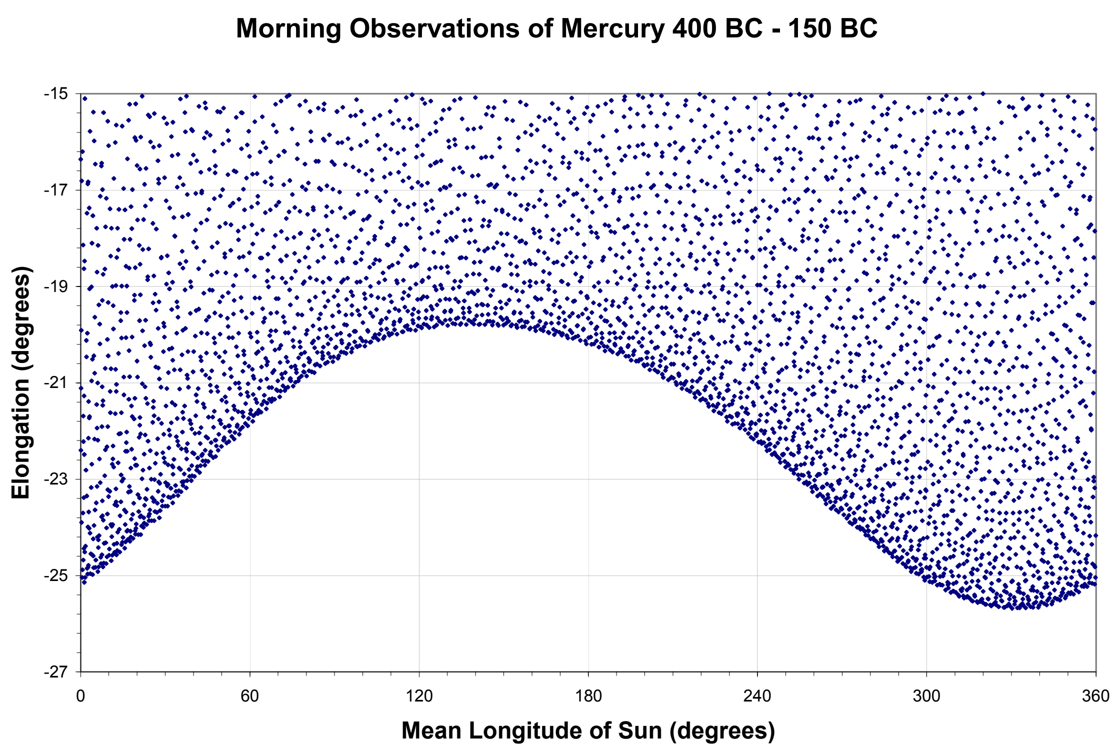



Figure 6.

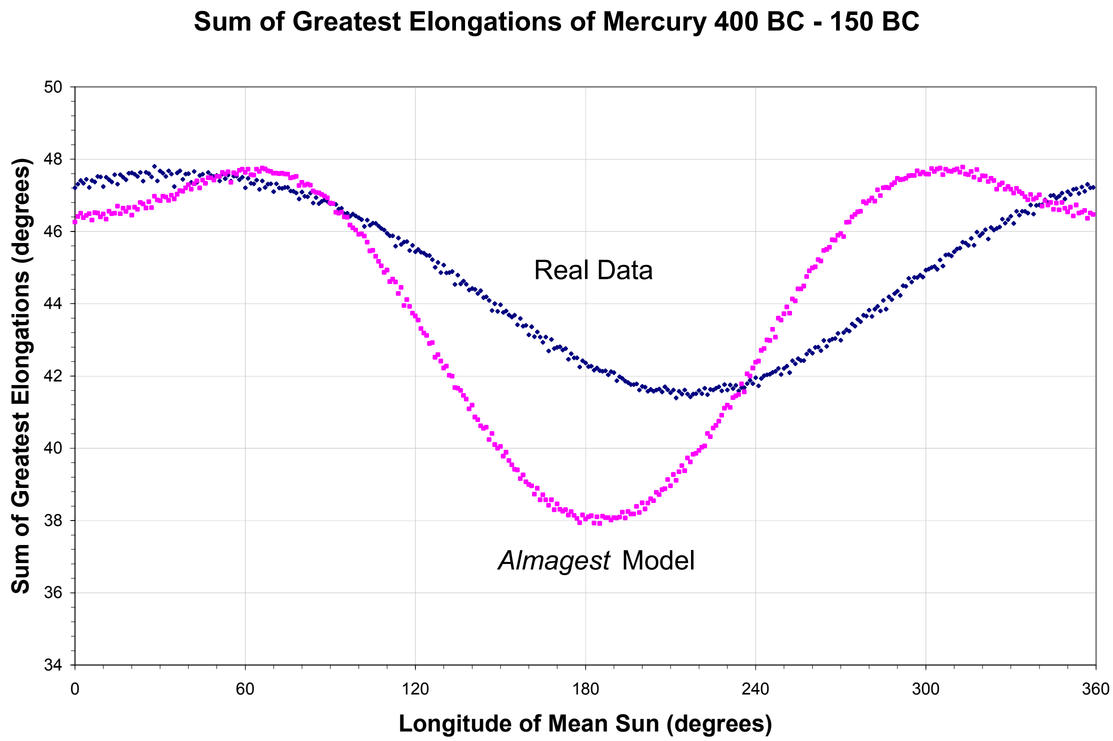



Figure 7.

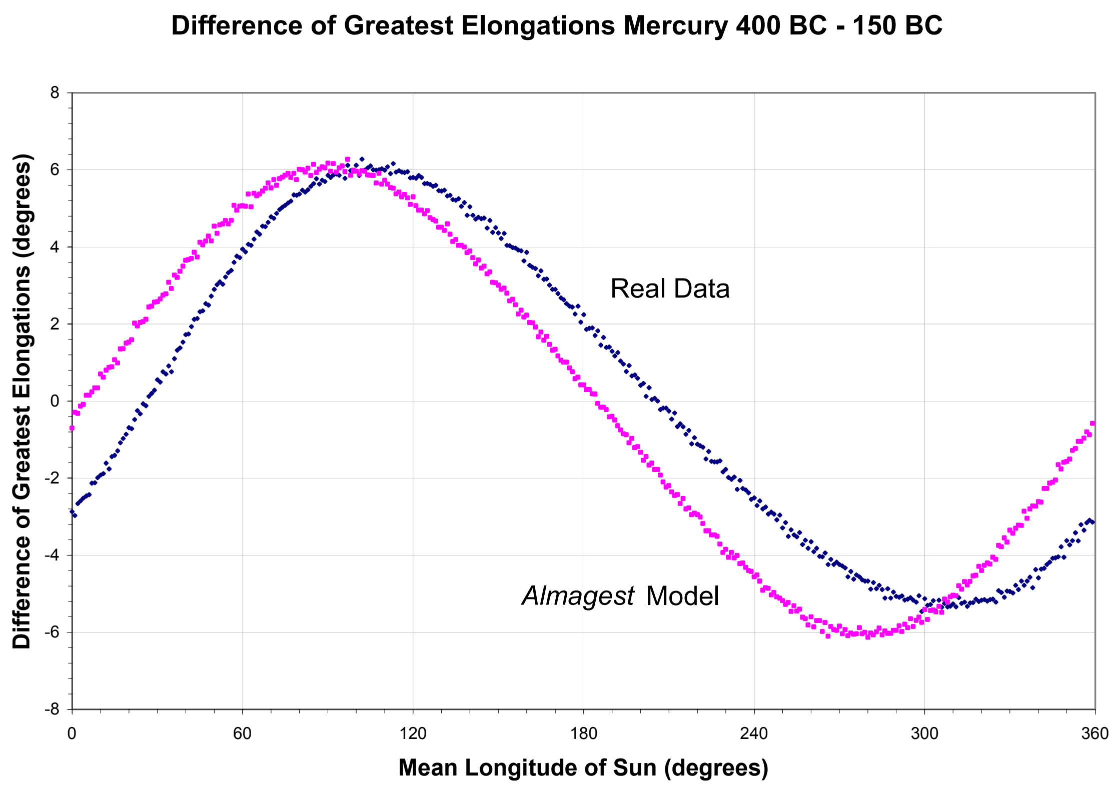



Figure 8.

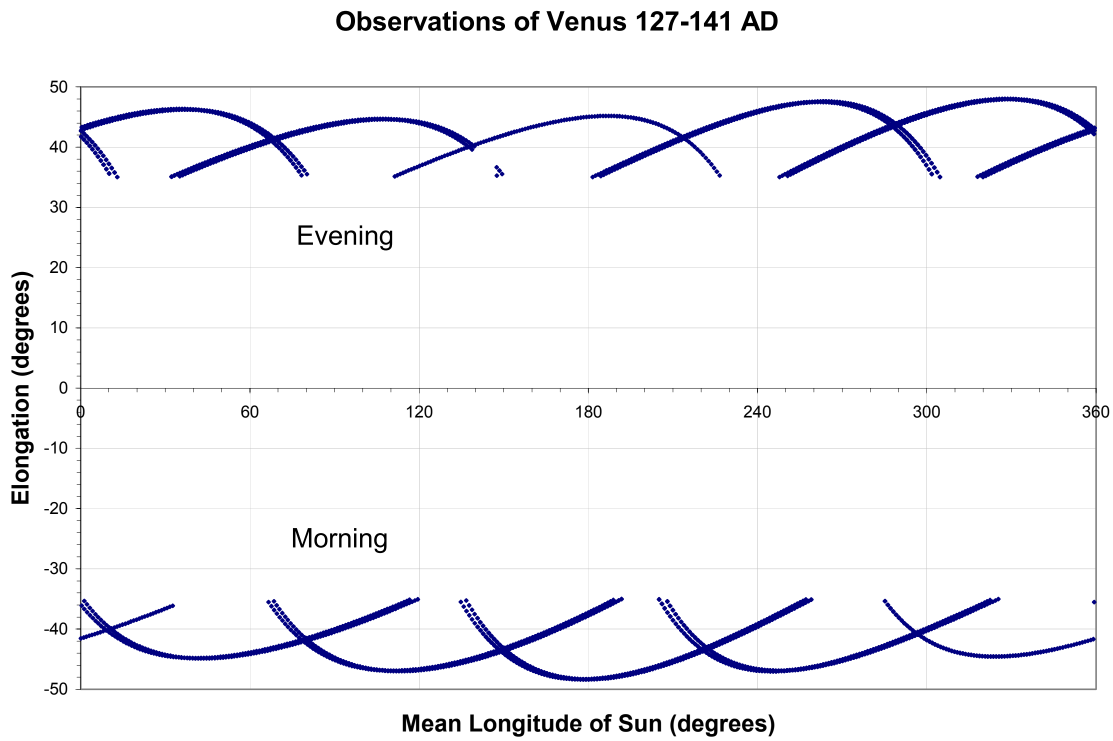



Figure 9.

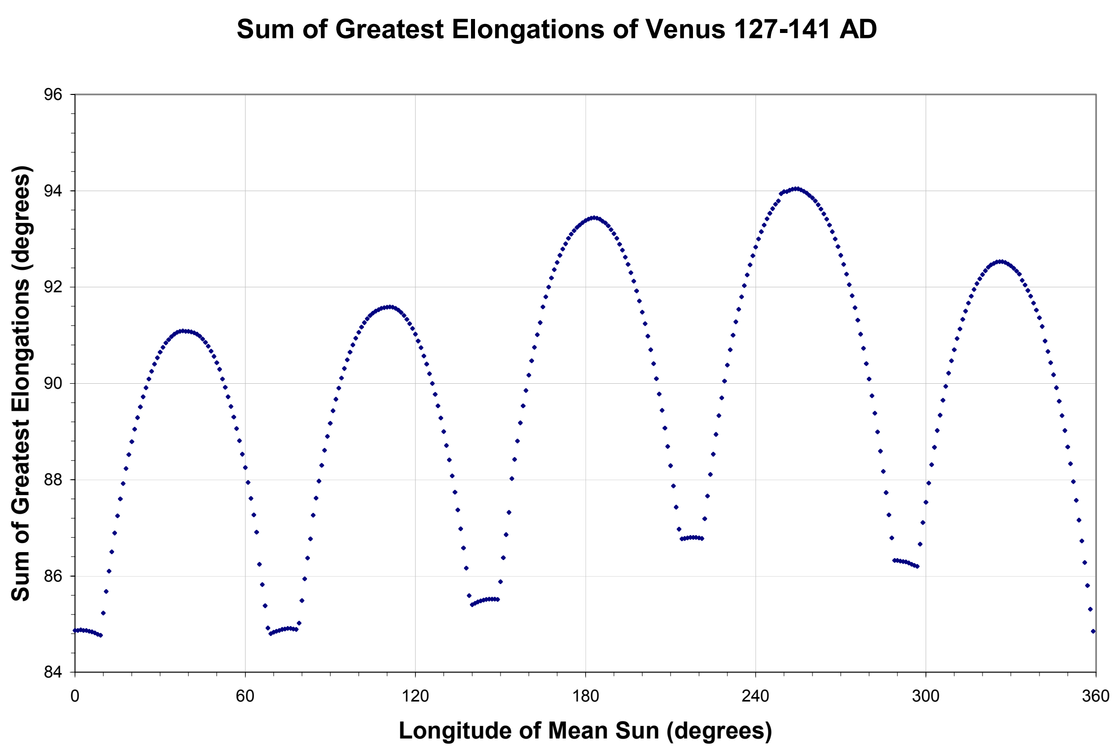



Figure 10.

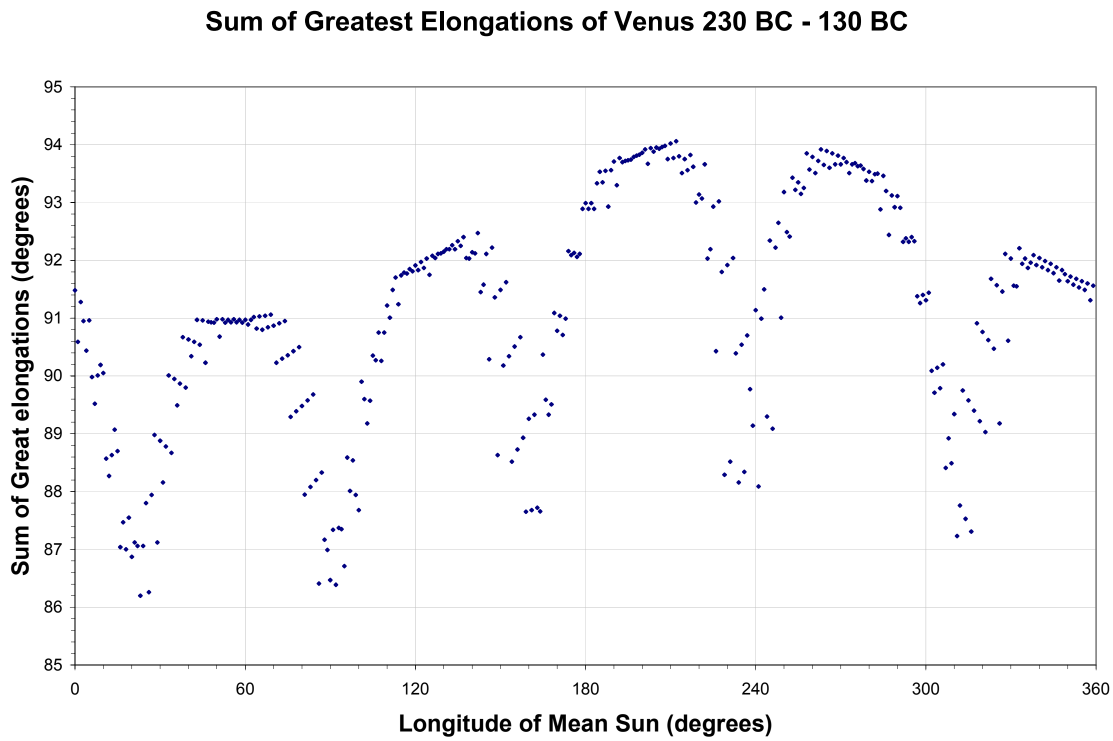



Figure 11.

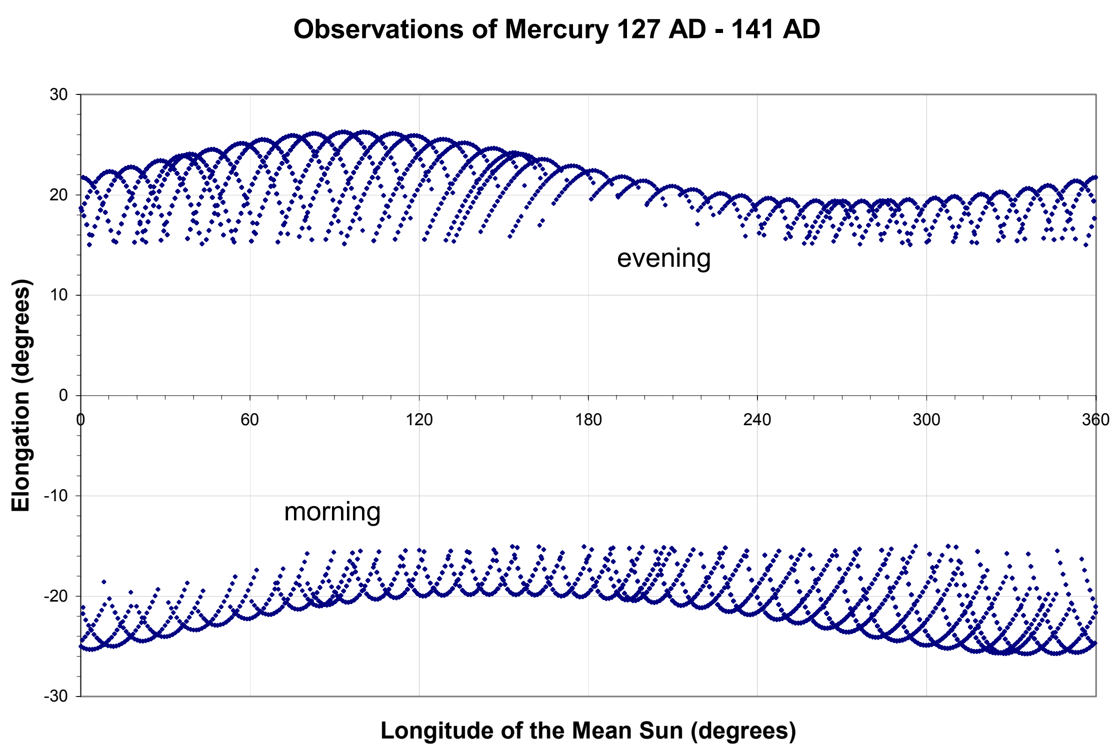



Figure 12.

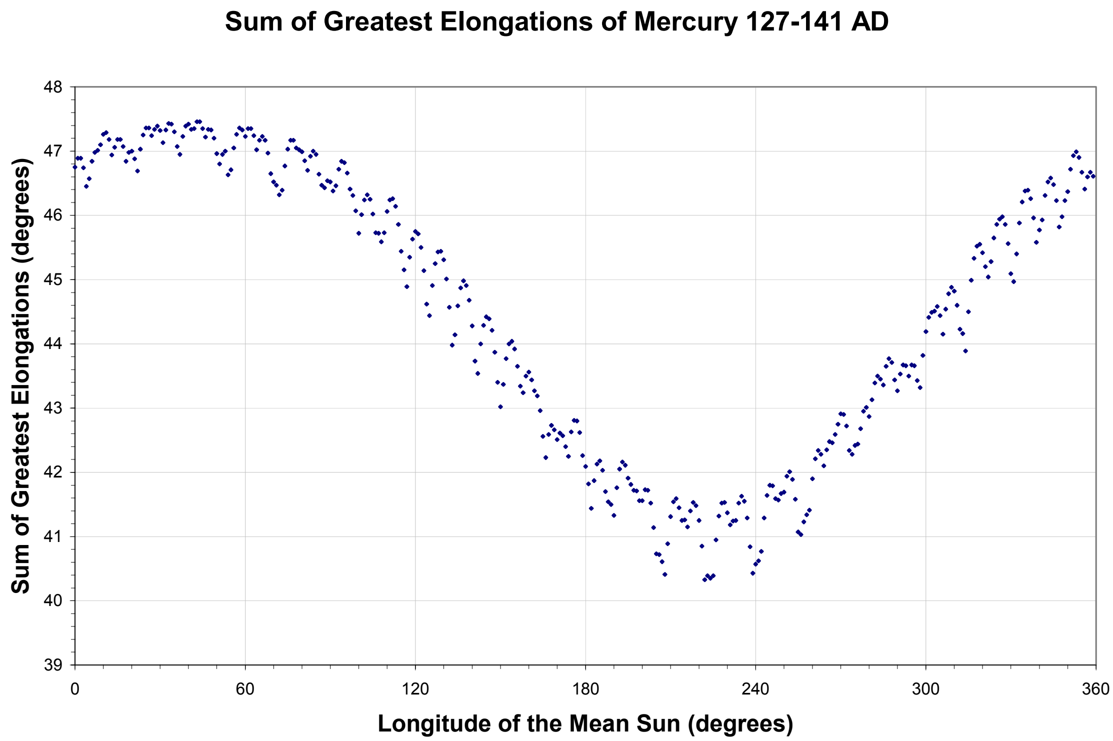



Figure 13.

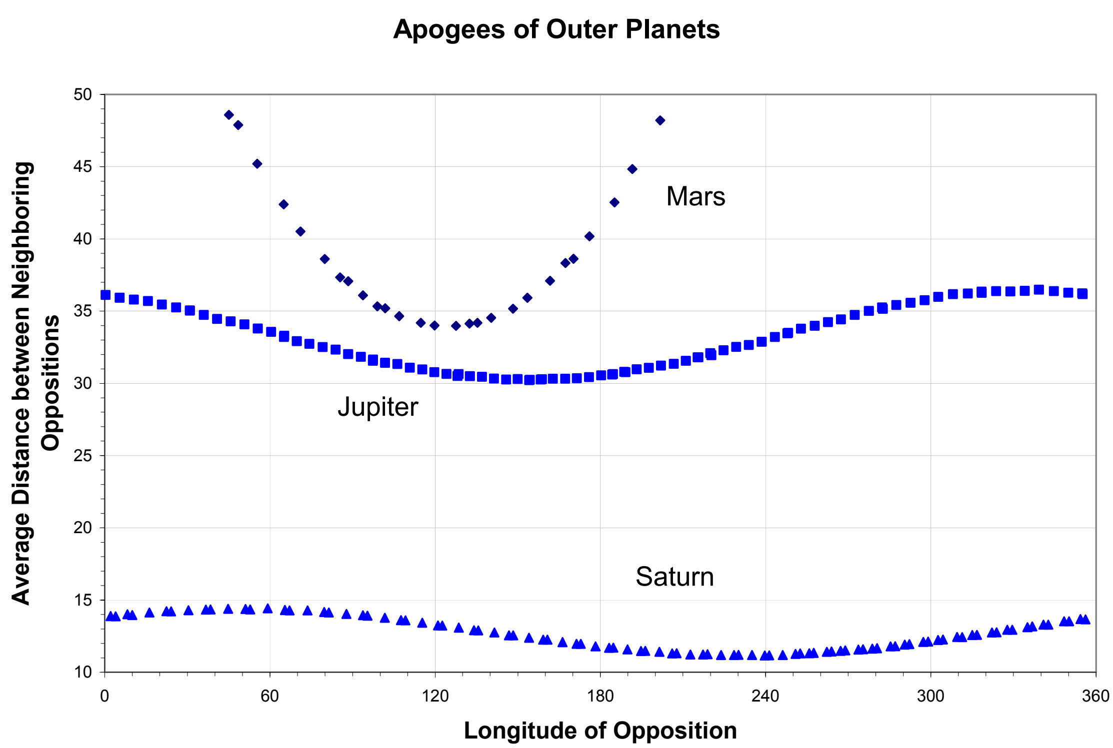
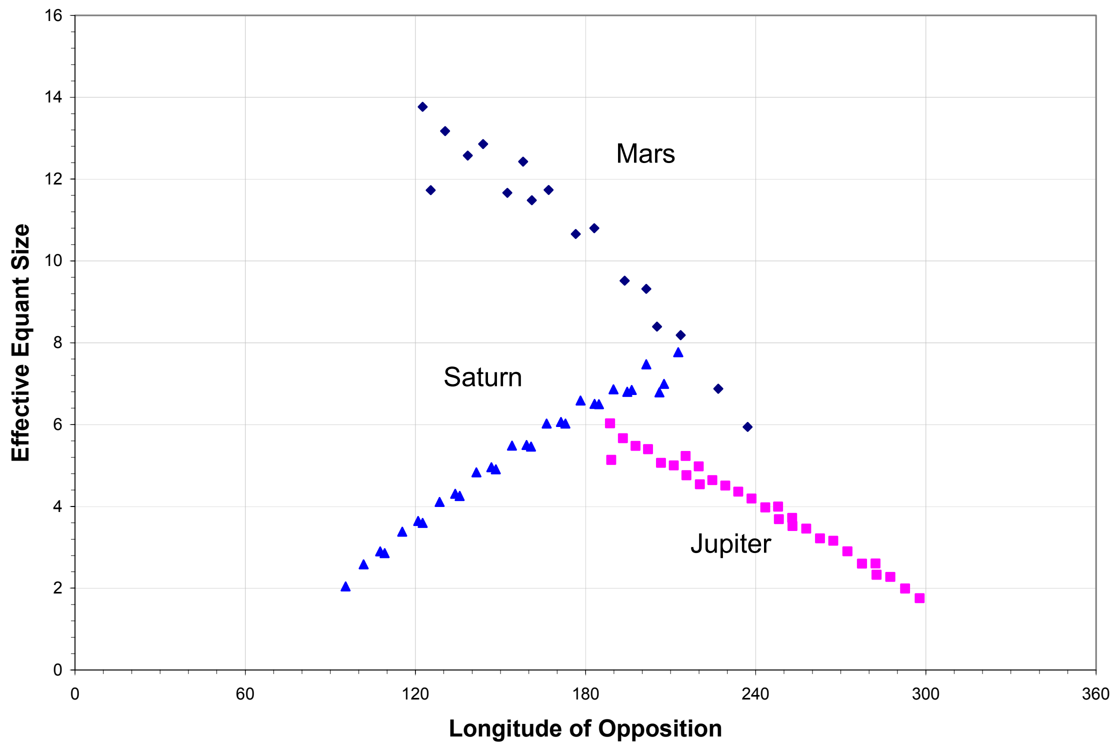



Figure 14.



16
14
12
10
8
6
4
2
0
Effective Equant Size
0
60
120
180
240
300
360
Longitude of Opposition
Mars
Saturn
Jupiter


Figure 15.

REFERENCES

[1] *Ptolemy's Almagest*, transl. by G. J. Toomer (London, 1984).

[2] C. Wilson, "The Inner Planets and the Keplerian revolution", *Centaurus*, 17 (1972) 205-248; R. R. Newton, *The crime of Claudius Ptolemy*, (Baltimore, 1977); N. M. Swerdlow, "Ptolemy's Theory of the Inferior Planets", *Journal for the history of astronomy*, 20 (1989) 29-60.

[3] Pierre Bretagnon and Jean-Louis Simon, *Planetary Programs and Tables from –4000 to +2800*, (Richmond, 1986).

[4] The greatest elongations are assembled using a simple tabular method, much as the ancient analyst might have done. Start with an empty table of 360 rows, one row for each rounded value of the longitude of the mean Sun. When an elongation is determined, look into the corresponding row of the table and enter the value of the elongation if it is larger than the value already in the table. One might also save the date and any other pertinent information.

[5] *Almagest* IX.2 refers to models consisting of eccentric circles, concentric circles carrying epicycles, and both combined. Also, the iterative process of finding the equant and apogee for the outer planets using three oppositions begins by assuming an eccentric deferent carrying an epicycle.

[6] This is discussed in detail in Wilson, *op. cit.* (ref. 2) p. 211.

[7] C. Wilson, *op. cit.* (ref. 2) p. 234.

[8] The data in these charts is collected daily.

[9] However, the discussion of this paragraph must be tempered by the recent demonstrations by Rawlins (in a heliocentric presentation) and Thurston (in an equivalent geocentric presentation) that a straightforward geometrical analysis of one trio of elongations can yield an estimate of the eccentricity (or equant), the apogee direction, and the radius of the epicycle.

[10] An early mention of this is in the *Almagest* itself, chapter XIII.8, and it was also mentioned by the Babylonians. The point is also sometimes raised in our time, e. g. in Owen Gingerich, "Ptolemy and the Maverick Motion of Mercury", *Sky & Telescope,* 66 (1983) 11-13.

[11] These are the visibility conditions used by Jean Meeus, *More Mathematical Astronomy Morsels,* (Richmond, 2002) p. 347.

[12] N. M. Swerdlow, *The Babylonian Theory of the Planets,* (Princeton, 1998), p. 107.

[13] A. J. Sachs and H. Hunger, "Astronomical Diaries and related Texts from Babylonia. I-III"*, Denkschriften / Österreichische Akademie der Wissenschaften, Philosophisch-Historische Klasse* 195 (1988), 210 (1989), 246 (1996).

[14] H. Hunger, "A 3456: eine Sammlung von Merkurbeobachtungen", in E. Leichty *et al., A Scientific Humanist, Studies in Memory of Abraham Sachs. Occasional Papers of the Samuel Noah Kramer Fund* 9, (Philadelphia, 1988).

[15] James Evans, *The History and Practice of Ancient Astronomy*, (New York, 1998), p. 362-368.

[16] James Evans, "On the Function and the Probable Origin of Ptolemy's Equant", *American Journal of Physics*, 52 (1984) 1080-1089.