The Very Early History of Trigonometry

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The early history of trigonometry, say for the time from Hipparchus through Ptolemy, is fairly well established, at least in broad outline (van Brummelen 2009). For these early astronomers plane trigonometry allowed the solution of an arbitrary right triangle, so that given either of the non-90° angles one could find the ratio of any two sides, or given a ratio of sides one could find all the angles. In addition the equivalent of the law of sines was known, although use infrequently, at least by Ptolemy. This skill was fully developed by the time Ptolemy wrote the *Almagest*, *ca* 150 CE (Toomer 1980), and he used it to solve a multitude of problems, some of them quite sophisticated, related to geometric models of astronomy. Ptolemy's sole tool for solving trigonometry problems was the chord: the length of the line that subtends an arc of arbitrary angle as seen from the center of a circle. Using a standard circle of radius 60, the *Almagest* gives a table of these chords for all angles between ½° and 180° in increments of ½°, and indeed Ptolemy gives a fairly detailed account of how one can compute such a table using the geometry theorems known in his time. Curiously, but not all that unusual for Ptolemy, it appears that some of the chord values in the *Almagest* were not in fact derived using the most powerful theorems that Ptolemy possessed (van Brummelen 1993, 46-73).

We also have evidence from Ptolemy that Hipparchus, working around 130 BCE, was able to solve similar trigonometry problems of about the same level of difficulty. For example, regarding finding the eccentricity and direction of apogee for the Sun's simple eccentric model, Ptolemy writes , Ptolemy writes in *Almagest* III 4:

> These problems have been solved by Hipparchus with great care. He assumes that the interval from spring equinox to summer solstice is 92½ days, and that the interval from summer solstice to autumn equinox is 92½ days, and then, with these observations as his sole data, shows that the line segment between the above-mentioned centres is approximately $\frac{1}{24}^{th}$ of the radius of the eccentre, and that the apogee is approximately 24½° in advance of the summer solstice.

The similar problem of finding the eccentricity and direction of apogee for the Moon's simple epicycle model is complicated by the moving lunar apogee. A glance at Figure 1 and a few moments consideration might give you some feel for the more advanced difficulty level of this particular problem. that Ptolemy explains in *Almagest* IV 6:

> In this first part of our demonstrations we shall use the methods of establishing the theorem which Hipparchus, as we see, used before us. We, too, using three lunar eclipses, shall derive the maximum difference from the mean motion and the epoch of the [moon's position] at the apogee, on the assumption that only this [first] anomaly is taken into account, and that it is produced by the epicyclic hypothesis.

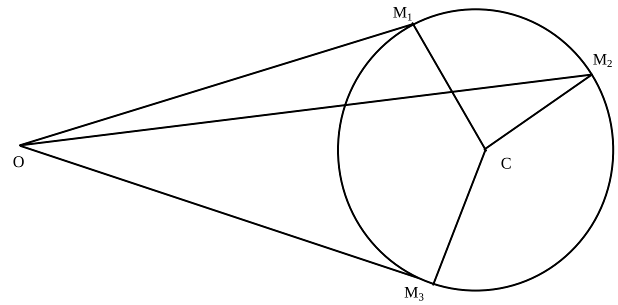

Figure 1. Consider a circle with center C and radius r. Let the distance $OC = R$. The angles M_1CM_2, M_2CM_3 and M_1OM_2, M_2OM_3 are given, and the problem is to find r/R. For a solution see *Almagest* IV 6 or Toomer 1973.

Finally, in *Almagest* IV 11 Ptolemy presents two trios of lunar eclipses that he says Hipparchus had used to determine the size of the first anomaly in lunar motion. Ptolemy gives just the results of Hipparchus' solutions, and from these we learn that while Hipparchus was certainly a capable user of trigonometry, he used a different set of numerical conventions than those used by Ptolemy. For example, while Ptolemy used a standard 360° degree circle with a radius of 60 parts, Hipparchus apparently specified the circumference of his circle as having 21,600 (= 360 x 60) parts, so that his diameter was about 6875 parts and his radius was about 3438 parts (Toomer 1973). We cannot, however, be sure whether Hipparchus used the same chord construct as Ptolemy, or perhaps just gave the ratio of side lengths corresponding to a set of angles. Nor can we be sure whether Hipparchus used a systematized table, or if he did, the angle increments of that table (Duke 2005).

One attempt to resolve these questions comes not from Greek or Roman sources, but from texts from ancient India that date from perhaps 400 – 600 CE. For many reasons, including the use of the circumference convention identical to that used by Hipparchus, and in spite of their appearance in India some six centuries after Hipparchus, it is has been proposed that these texts reflect a Greco-Roman tradition that is pre-Ptolemaic and largely otherwise unknown to us (Neugebauer 1956, Pingree 1976,1978, van der Waerden 1961). These proposals have so far eluded definitive confirmation (and neither have any effective refutations appeared), but if they are true for the parts involving trigonometry, then it would seem plausible that Hipparchus' working set of tools included tables with 23 (non-trivial) entries of side ratios in angular increments of 3¾°, corresponding to chords in increments of 7½°, for we find exactly such tables in many Indian texts, always embedded in astronomical material that is extremely similar to early Greek astronomy.

We might be able to understand Hipparchus' use of trigonometry somewhat better if we had a little more idea how it was developed. There is a Greek source that might well be helpful in this regard, namely Archimedes' *Measurement of a Circle* (Heath 1897). Archimedes' mathematical methods in this paper are well-known: he uses the bounds

$$\tfrac{265}{153} < \sqrt{3} < \tfrac{1351}{780}$$

on $\sqrt{3}$ and then alternately circumscribes and inscribes a set of regular polygons around a circle, ultimately computing the ratio of the circumference of 96-sided polygons inside and outside the circle to the diameter of the circle, thus establishing bounds on π as

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}$$

What Archimedes actually computes in both cases (circumscribing and inscribing), however, are the ratios of the lengths of sides for a series of right triangles with smallest interior angle 30°, 15°, 7½°, 3¾°, (and partially $1\tfrac{7}{8}^{\circ}$), and so except for normalization many of the entries for the tables used in India and perhaps also by Hipparchus are computed in Archimedes' text, and all the entries are easily found using Archimedes' method.

Thus, denoting the opposite side, the adjacent side, and the hypotenuse by *a*, *b,* and *c* Archimedes finds for the circumscribed sequence of right triangles ratios of the following values:

	a	*b*	*c*
30°	153	265	306
15°	153	571	591 1/8
7½°	153	1162 1/8	1172 1/8
3¾°	153	2334 3/8	2339 3/8

The entries in the first row result from Archimedes' lower bound on $\sqrt{3}$, while the entries in row *i*+1 follow from those in row *i* using Archimedes' algorithm:

$$a_{i+1} = a_i$$
$$b_{i+1} = b_i + c_i$$
$$c_{i+1} = \sqrt{a_{i+1}^2 + b_{i+1}^2}$$

The ratios for the complementary angles 60°, 75°, 82½°, and 86¼° are trivially obtained by interchanging columns *a* and *b*, and we now have the ratios for eight of the 23 non-trivial angles in the sequence. We may get an additional eight values by applying Archimedes' algorithm to the angles 82 ½ °, yielding the table entries for 41¼° and 48¾°, to the angle 75°, yielding the entries for 37½°, 52½°, 18¾°, and 71¼°, and to the angle 52½ °, yielding the entries for 26¼° and 63¾°. Thus we get:

	a	*b*	*c*
41¼°	1162 1/8	1324 7/8	1762 3/8
37½°	571	744	937 7/8
18¾°	571	1682	1776 1/4
26¼°	744	1508 7/8	1682 3/8

and the ratios for the complementary angles again come from interchanging *a* and *b*.

Thus 16 of the 23 table entries are immediately available directly from Archimedes' text. To get the remaining seven entries it is necessary to repeat Archimedes' analysis beginning from a 45° right triangle and bounds on $\sqrt{2}$. If Archimedes used the bounds

$$\tfrac{1393}{985} < \sqrt{2} < \tfrac{577}{408}$$

then one would find for the sequence of circumscribed triangles ratios of the following values:

	a	*b*	*c*
45°	985	985	1393
22½°	985	2378	2573 7/8
11¼°	985	4951 7/8	5049
33¾°	2378	3558 6/8	4280 1/8

and the ratios for the complimentary angles 67½°, 78¾°, 56¼° again follow from interchanging *a* and *b*.

The analysis of the inscribed triangles follows the same algorithm but instead begins with the upper bounds on $\sqrt{3}$ and $\sqrt{2}$. The resulting bounds on the ratios are so close that for all practical purposes – let us remember, these are used for analysis of measured astronomical angles, and we use linear interpolation for untabulated angles – we can use either set, or their average, with no appreciable difference in results. Here is the entire set of entries:

	circumscribed		inscribed		circumscribed	inscribed
Angle	*a*	*c*	*a*	*c*	Base 3438	Base 3438
3 6/8	153	2339 3/8	780	11926	225	225
7 4/8	153	1172 1/8	780	5975 7/8	449	449
11 2/8	985	5049	408	2091 3/8	671	671
15	153	591 1/8	780	3013 6/8	890	890
18 6/8	571	1776 2/8	2911	9056 1/8	1105	1105
22 4/8	985	2573 7/8	408	1066 1/8	1316	1316
26 2/8	744	1682 3/8	3793 6/8	8577 3/8	1520	1520
30	153	306	780	1560	1719	1719
33 6/8	2378	4280 1/8	985	1773	1910	1910
37 4/8	571	937 7/8	2911	4781 7/8	2093	2093
41 2/8	1162 1/8	1762 3/8	5924 6/8	8985 6/8	2267	2267
45	985	1393	408	577	2431	2431
48 6/8	1324 7/8	1762 3/8	6755 7/8	8985 6/8	2584	2584
52 4/8	744	937 7/8	3793 6/8	4781 7/8	2727	2727
56 2/8	3558 6/8	4280 1/8	1474 1/8	1773	2858	2858
60	265	306	1351	1560	2977	2977
63 6/8	1508 7/8	1682 3/8	7692 7/8	8577 3/8	3083	3083
67 4/8	2378	2573 7/8	985	1066 1/8	3176	3176
71 2/8	1682	1776 1/8	8575 4/8	9056 1/8	3255	3255
75	571	591 1/8	2911	3013 6/8	3320	3320
78 6/8	4951 7/8	5049	2051 1/8	2091 3/8	3371	3371
82 4/8	1162 1/8	1172 1/8	5924 6/8	5975 7/8	3408	3408
86 2/8	2334 3/8	2339 3/8	11900 4/8	11926	3430	3430

In the table above, for each angle in col. 1 cols. 2–3 and cols. 4–5 give the lengths of the opposite side and the hypotenuse for the circumscribed and inscribed triangles, respectively, in Archimedes’ method. Cols. 6 and 7 give the rounded length of the opposite side assuming the hypotenuse has length 3438 parts, corresponding to a circumference of 21,600 parts. Note that for all 23 angles the ratios for each angle are identical to the level of approximation used.

Therefore, we see that using Archimedes’ method, and in many cases the very numbers that appear in his text, anyone could have assembled the table in increments of 3¾° that was used in India and might have been used by Hipparchus. The two steps needed to go beyond Archimedes are (a) a normalization convention, and (b) an interpolation scheme, and there seems no reason to doubt that any competent mathematician of the time would have the slightest trouble dealing with either issue. We are certainly in no position to say that Archimedes himself constructed the table, or who in the century between Archimedes and Hipparchus did it, but it is clear that by the time of Archimedes’ paper all the needed tools and results were in place, except possibly for the motivation to actually organize the table.

We can, in fact, go even farther back into the very early history of trigonometry by considering Aristarchus' *On Sizes and Distances* (Heath 1913), and we shall see that a plausible case can be made that his paper could easily have been the inspiration for Archimedes' paper. The problem Aristarchus posed was to find the ratio of the distance of the Earth to the Moon to the distance of the Earth to the Sun. He solved this problem by assuming that when that the Moon is at quadrature, meaning it appears half-illuminated from Earth and so the angle Sun-Moon-Earth is 90°, the Sun-Moon elongation is 87°, and so the Earth-Moon elongation as seen from the Sun would be 3°. Thus his problem is solved if he can estimate the ratio of opposite side to hypotenuse for a right triangle with an angle of 3°, or simply what we call sin 3°. In addition, for other problems in the same paper Aristarchus also needed to estimate sin 1° and cos 1°.

Aristarchus proceeded to solve this problem is a way that is very similar to, but not as systematic as, the method used by Archimedes. By considering circumscribed (Fig. 2) and inscribed triangles (Fig 3) and assuming a bound on $\sqrt{2}$ Aristarchus effectively establishes bounds on sin 3° as

$$\tfrac{1}{20} < \sin 3^{\circ} < \tfrac{1}{18}$$

and, although he does not mention it, this also establishes bounds on π as

$$3 < \pi < 3\tfrac{1}{3}$$

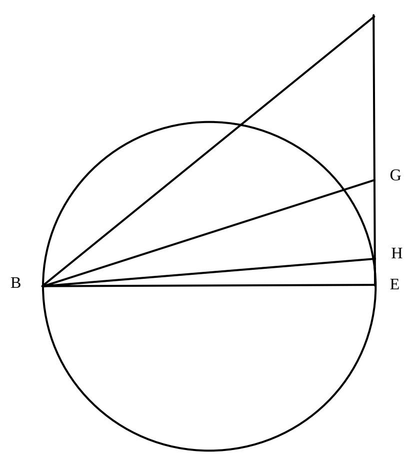

Figure 2. BE is a diameter of the circle, angle EBF is 45°, angle EBG is 22½°, and angle EBH is 3° (not to scale). Since EBG/EBH = 15/2 then GE/EH > 15/2. Since FG/GE = $\sqrt{2}$ > 7/5 then FE/EG > 12/5 = 36/15 and so FE/EH > (36/15)(15/2) = 18/1.

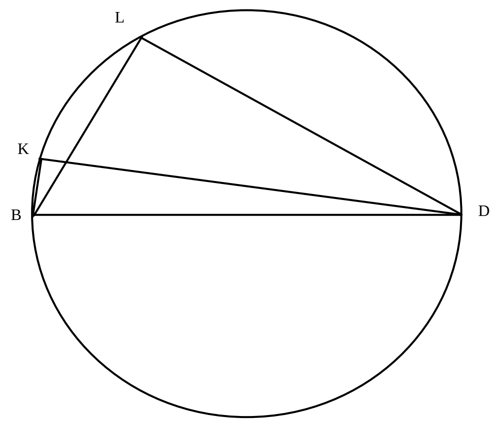

Figure 3. BD is a diameter of the circle, angle BDL = 30°, and angle BDK = 3° (not to scale). Since arc BL = 60° and arc BK = 6° then BL/BK < 10/1. Since BD = 2 BL then BD/BK < 20/1.

Later, in Propositions 11 and 12 Aristarchus proves using similar methods that

$$\tfrac{1}{60} < \sin 1^\circ < \tfrac{1}{45}$$

and

$$\tfrac{89}{90} < \cos 1^\circ < 1$$

always understanding, of course, that what we write as sine and cosine was to Aristarchus a ratio of sides in a right triangle. None of these bounds are particularly tight, and it is difficult to know if this was the best Aristarchus could do, or whether it was simply adequate for his purposes, which is apparently the case in any event.

The similarities between Aristarchus' and Archimedes' methods are clear: both assume bounds on a small irrational number, and hence effectively on the value of sin α for some relatively large angle, 60° or 45°, and through a sequence of circumscribed and inscribed triangles on a circle establish bounds on a target small angle, 3° for Aristarchus and $1\tfrac{7}{8}^\circ$ for Archimedes. Archimedes clearly realizes that this established bounds on π; Aristarchus may or may not have realized it, or might have not considered his bounds interesting enough to mention. Both Aristarchus and Archimedes are focused firmly on the relations between angles and ratios of sides in right triangles, neither ever using anything related to the chord construct used by Ptolemy. We know that Archimedes and Aristarchus exchanged correspondence, and we know that Archimedes was well aware of Aristarchus' work on the Earth–Moon–Sun distance problem. Indeed, Archimedes tells us that his own father also worked on the problem. In any case the parallels in the two calculations are quite striking, and it is not hard to imagine that Aristarchus' calculation could have been the inspiration behind Archimedes' calculation.

Coupled with the fact that the sin and not the chord is used also in the Indian texts, this suggests that the chord was introduced later rather than sooner, and certainly offers no encouragement to anyone claiming that Hipparchus used chords or that the sine was invented in India as an 'improvement' over the chord.

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