An Early Use of the Chain Rule

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One of the most useful tools we learned when we were young is the chain rule of differential calculus: if $q(\alpha)$ is a function of α, and $\alpha(t)$ is a function of t, then the rate of change of q with respect to t is

$$\frac{dq}{dt} = \frac{dq}{d\alpha} \cdot \frac{d\alpha}{dt}$$

In the special case that $a(t)$ is linear in t, so $\alpha(t) = \alpha_0 + \omega_a (t - t_0)$, this becomes

$$\frac{dq}{dt} = \frac{dq}{d\alpha} \omega_a$$

If $q(\alpha)$ is a complicated function of α, for example

$$q(\alpha) = \tan^{-1}\left(\frac{-e \sin \alpha}{R + e \cos \alpha} \right)$$

then the computation of $dq/d\alpha$ is not necessarily easy. In this case

$$\frac{dq}{d\alpha} = \frac{-e / R \cos \alpha - (e / R)^2}{1 + 2e / R \cos \alpha + (e / R)^2}$$

so when e/R is small we have simply

$$\frac{dq}{d\alpha} \simeq -\frac{e}{R} \cos \alpha$$

In cases like this a practical alternative is to tabulate $q(\alpha)$ at small intervals $\Delta\alpha$ and then estimate $dq/d\alpha$ as a ratio of finite differences:

$$\frac{dq(\alpha)}{d\alpha} \simeq \frac{q(\alpha + \Delta\alpha) - q(\alpha)}{\Delta\alpha}$$

This particular function $q(\alpha)$ in our example is, of course, the equation of center for the simple eccentric (or, equivalently, epicycle) model used by Hipparchus and later Ptolemy for the Sun and the Moon (at syzygy), and it connects the mean longitude $\bar{\lambda}$ and true longitude λ according to

$$\lambda = \bar{\lambda} + q(\alpha)$$

where $\alpha = \bar{\lambda} - A$ and A is the longitude of apogee. As we shall see, Ptolemy very clearly knew that the rate of change with time of the true longitude λ is

$$\frac{d\lambda}{dt} = \omega_t + \omega_a \frac{dq}{d\alpha}$$

where ω_t and ω_a are the mean motion of the Moon in longitude and anomaly. Actually proving the chain rule is straightforward enough, but not entirely trivial, although perhaps in this simple case it might be guessed by dimensional analysis. As is often the case, Ptolemy gives no hint of how he came to know it.

It is, I think, not as widely appreciated as it might be that the result just given appears in Ptolemy's *Almagest*, not once but twice, and so was known at least as early the 2nd century CE, and very probably was known to Hipparchus in the 2nd century BCE, therefore nearly two millennia before the development of differential calculus (for standard treatments see, e.g. Neugebauer 1975, 122-124, 190-206 or Pedersen 1974, 225-226).

The first occurrence of this result is found in *Almagest* VI 4. Ptolemy has just completed explaining how to compute the time $\bar{t}$ of some mean syzygy – a conjunction or opposition of the Sun and Moon in mean longitude – using their known mean motions and epoch positions in mean longitude and anomaly, and is ready to show how to estimate the time $t = \bar{t} + \delta t$ of the corresponding true syzygy. Therefore let us consider the case of a mean conjunction at some time $\bar{t}$, so that

$$\bar{\lambda}_S(\bar{t}) = \bar{\lambda}_M(\bar{t})$$

and work out what Ptolemy would do if he knew calculus.

Since we know the mean anomalies $\alpha_S(\bar{t})$ and $\alpha_M(\bar{t})$ at time $\bar{t}$ we can also compute the equations $q_S(\alpha(\bar{t}))$ and $q_M(\alpha(\bar{t}))$. At time t of true syzygy we have

$$\bar{\lambda}_S(t) + q_S(\alpha_S(t)) = \bar{\lambda}_M(t) + q_M(\alpha_M(t))$$

(with, of course, the addition of 180° on one side of the equation in the case of an opposition). Since the mean longitudes vary linearly in time we have simply

$$\bar{\lambda}_M(t) = \bar{\lambda}_M(\bar{t} + \delta t) = \bar{\lambda}_M(\bar{t}) + \omega_t \delta t$$
$$\bar{\lambda}_S(t) = \bar{\lambda}_S(\bar{t} + \delta t) = \bar{\lambda}_S(\bar{t}) + \omega_S \delta t$$

where ω_S is the mean motion of the Sun, so that

$$\bar{\lambda}_M(t) - \bar{\lambda}_S(t) = (\omega_t - \omega_S)\delta t = \eta \delta t = q_S(\alpha_S(t)) - q_M(\alpha_M(t))$$

Furthermore, since δt is small compared to the orbital period of the Moon, and even more so the Sun, we have

$$\begin{aligned} q_M(\alpha_M(t)) &= q_M(\alpha_M(\bar{t})) + \delta t \left.\frac{dq_M}{dt}\right|_{t=\bar{t}} + O(\delta t^2) \\ &\simeq q_M(\alpha_M(\bar{t})) + \omega_a \delta t \left.\frac{dq_M}{d\alpha}\right|_{t=\bar{t}} \\ q_S(\alpha_S(t)) &= q_S(\alpha_S(\bar{t})) + \delta t \left.\frac{dq_S}{dt}\right|_{t=\bar{t}} + O(\delta t^2) \\ &\simeq q_S(\alpha_S(\bar{t})) + \omega_S \delta t \left.\frac{dq_S}{d\alpha}\right|_{t=\bar{t}} \end{aligned}$$

noting that for the standard solar model of Hipparchus and Ptolemy the mean motions in longitude and anomaly of the Sun are equal since the solar apogee is tropically fixed.

Combining these and solving for δt gives

$$\delta t = \frac{q_S(\alpha_S(\bar{t})) - q_M(\alpha_M(\bar{t}))}{\eta + \omega_a \left.\frac{dq_M}{d\alpha_M}\right|_{t=\bar{t}} - \omega_S \left.\frac{dq_S}{d\alpha_S}\right|_{t=\bar{t}}}$$

Ptolemy, of course, does not know how to do a Taylor expansion approximation, but the result he gives is uncannily similar. First he instructs us to estimate the true distance between the Sun and Moon at mean syzygy, which we see from the above is

$$q_S(\alpha_S(\bar{t})) - q_M(\alpha_M(\bar{t}))$$

He then says to multiply this by $^{13}\!/_{12}$ and to divide that result by the Moon's true speed, which he estimates as

$$0;32,56^{\circ/hr} - 0;32,40^{\circ/hr}(\mathrm{q}(\alpha + 1^{\circ}) - q(\alpha))$$

where $0;13,56^{\circ/\mathrm{hr}}$ is the Moon's mean motion in longitude ω_t expressed in degrees per equinoctial hour, and similarly $0;32,40^{\circ/\mathrm{hr}}$ is the hourly mean motion in anomaly. Note also that

$$q(\alpha + 1^{\circ}) - q(\alpha) = \frac{q(\alpha + 1^{\circ}) - q(\alpha)}{1^{\circ}} = \left.\frac{\Delta q}{\Delta \alpha}\right|_{t=\bar{t}}$$

so Ptolemy has estimated $dq/d\alpha$ with a finite difference approximation, and furthermore chosen an interval $\Delta\alpha = 1°$ that, at first sight, cleverly avoids an otherwise necessary division operation.

So in the end his estimate of the correction δt to the mean time $\bar{t}$ is, in units of equinoctial hours,

$$\delta t = \frac{q_S(\alpha_S(\bar{t})) - q_M(\alpha_M(\bar{t}))}{\frac{12}{13}\left(0;32,56° + 0;32,40° \left.\frac{dq_M}{d\alpha_M}\right|_{t=\bar{t}}\right)}$$

which compares very closely to the more exact result derived above, the only differences being that he has two approximations in the denominator: first, he gives

$$\frac{12}{13} \times 0;32,56 = 0;30,24$$

which is a good approximation to η = 0;30,8, and second he neglects the term proportional to $dq_S/d\alpha_S$ which is an order of magnitude smaller than the already small (compared to 0;32,56) derivative of the Moon's anomalistic equation of center.

Although Ptolemy's scheme of estimating $dq/d\alpha \simeq q(\alpha + 1°) - q(\alpha)$ is certainly one option, it is not necessarily the best option when the task is to make the estimate using a table of $q(\alpha)$ values. For one reason, it requires two table interpolations. Yet these can be easily avoided if the instructions are instead to find the interval in which α lies, i.e. find α_i and α_{i+1} such that $\alpha_i \le \alpha < \alpha_{i+1}$ (which can be done by inspection), and then estimate $dq/d\alpha$ using

$$\frac{dq(\alpha)}{d\alpha} = \frac{q(\alpha_{i+1}) - q(\alpha_i)}{\alpha_{i+1} - \alpha_i}$$

which, given the piecewise linearity of the table, is about the best estimate you can make in any case without resorting to a higher order interpolations scheme. Furthermore, the quotients on the right hand side of the above equation could all be precomputed and included in the table and would be useful for all table interpolations, but that is not done in the *Almagest*. Thus, the procedure that Ptolemy describes would make a lot more sense, especially in terms of computational efficiency, if the table was compiled with an interval of 1° in the variable α. Strabo tells us that for geography Hipparchus did compile length of the longest day at intervals of 1° in terrestrial latitude, so it would not be too surprising if Hipparchus had 1° tables for lunar, and for that matter, solar anomaly.

Ptolemy goes on to estimate how close to the nodes the Moon has to be before an eclipse is even possible. For lunar eclipses this is straightforward, but for solar eclipses a rather involved calculation involving lunar parallax is required, lunar parallax having already been analyzed in detail in *Almagest* V 17–19. Ptolemy then discusses the allowed intervals (in months) between lunar and solar eclipses. Besides the common six month interval, it turns out that lunar eclipses can also occur at five month, but not seven month, intervals, and solar eclipses can occur at not only both five and seven month intervals, but also at one month intervals, provided the observers are at widely different locations, including being in different (north and south) hemispheres.

Related to all this is a passage in Pliny's *Natural History*, written *ca*. 70 CE, which says

> It was discovered two hundred years ago, by the sagacity of Hipparchus, that the moon is sometimes eclipsed after an interval of five months, and the sun after an interval of seven; also, that he becomes invisible, while above the horizon, twice in every thirty days, but that this is seen in different places at different times.

For Hipparchus to know all this, and in particular the part about solar eclipses at one month intervals, requires that he had a significant amount of computational skill, including a reasonable command of lunar parallax. Indeed, Ptolemy tells us that Hipparchus wrote two books on parallax. Therefore it is hardly a stretch to presume, with Neugebauer 1975, 129 and Pedersen 1974, 204, that Hipparchus already knew the eclipse material reported by Ptolemy in the *Almagest*, including the use of the chain rule discussed above.

The second occurrence of the use of the chain rule is in *Almagest* VII 2 concerning retrograde motion. Ptolemy begins by recalling Apollonius' treatment (from perhaps 180 BCE) of the simple epicycle model, in which the distance from the Earth to the epicycle center is constant. The ratio of a particular pair of geometric distances is, according to Apollonius' theorem, equal to the ratio of the speed ω_t of the epicycle center to the speed ω_a of the planet on the epicycle, both of which are constant in the simple model. However, in the case of the more complicated *Almagest* planetary models – the equant for Saturn, Jupiter, Mars, and Venus and the crank mechanism for Mercury – the relevant ratio is between the true speeds v_t and v_a as observed from Earth, which are not constant, and this once again involves using the chain rule, just as above:

$$\frac{d\lambda_t / dt}{d\lambda_a / dt} = \frac{\omega_t + \dfrac{dq}{dt}}{\omega_a + \dfrac{dq}{dt}} = \frac{\omega_t + \omega_t' \dfrac{dq}{d\alpha}}{\omega_a + \omega_t' \dfrac{dq}{d\alpha}}$$

where ω_t' is ω_t diminished by $1^{\circ/\mathrm{cy}}$ to account for the sidereally fixed apogees in the *Almagest* planetary models. In this case Ptolemy does not actually explain how to compute the numerical derivatives for $dq/d\alpha$, but the numerical values he gives for each planet confirm that he was using the tables of mean anomaly in *Almagest* XI 11, or something pretty close to them.

Returning now to eclipses, the natural question to wonder about is whether this careful estimate of the instantaneous speed is worth the effort? For example, how much difference would it make in eclipse predictions if in the calculations the mean speed η was used instead of the accurately calculated speed? In order to investigate this questions I have computed, using the *Almagest* rules, all 977 lunar eclipses from –746 to –130.

The speed is used two ways. First, it is used to compute the difference in time between mean and true conjunction, the eclipse being taken to occur at true conjunction rather than at minimum distance from the shadow center. This latter approximation is a good one, the time difference between true conjunction and minimum distance averaging less than 2 minutes and never exceeding 6 minutes, no matter which speed, mean or instantaneous, is used. On the other hand, the estimates of the actual time of true conjunction vary by about 19 minutes on average, and for about 40% of lunar eclipses the time difference exceeds 20 minutes, with a maximum difference of about 48 minutes.

Second, the speed is used to compute the duration of partial and total eclipse. Considering just partial eclipses, which are probably the easiest to time and show the largest effect in any event, the average difference in computed duration is about 12 minutes. and for about 14% of lunar eclipses the difference of computed duration of partial eclipse time interval exceeds 20 minutes, with a maximum difference of about 41 minutes. The differences that exceed 20 minutes arise when the eclipses have low magnitude, so that a relatively small change in the latitude of the Moon can result in a relatively large change in the path length needed to cross the shadow.

Altogether then, it seems reasonable to me that these differences in predicted absolute time and duration of lunar eclipses, while not exactly dramatic, are large enough to suggest a motivation for the ancient astronomer to compute the times using the instantaneous rather than the mean speed.

All of this by no means implies that differential calculus as we know it was understood by ancient mathematicians, but it does show that when they needed to solve a special problem, such as the one above, they were in some cases able to do it.

References

Neugebauer 1975. *A History of Ancient Mathematical Astronomy* (3 vols), Berlin.

Pedersen 1974. *A Survey of the Almagest*, Odense (reprinted by Springer in 2010 with annotation and new commentary by Alexander Jones).